COMBINATORIAL HOPF ALGEBRAS AND K-HOMOLOGY OF GRASSMANIANS

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ABSTRACT. Motivated by work of Buch on set-valued tableaux in relation to the K-theory of the Grassmannian, we study six combinatorial Hopf algebras. These Hopf algebras can be thought of as K-theoretic analogues of the by now classical "square" of Hopf algebras consisting of symmetric functions, quasisymmetric functions, noncommutative symmetric functions and the Malvenuto-Reutenauer Hopf algebra of permutations. In addition, we develop a theory of set-valued P-partitions and study three new families of symmetric functions which are weight generating functions of reverse plane partitions, weak set-valued tableaux and valued-set tableaux.

1. INTRODUCTION

The Hopf algebra Sym of symmetric functions [17], the Hopf algebra QSym of quasisymmetric functions [7], the Hopf algebra NSym of noncommutative symmetric functions [6] and the Malvenuto-Reutenauer Hopf algebra MR of permutations [13] can be arranged in the following diagram:

$$(1) \qquad \begin{array}{ccc} \text{Sym} & \twoheadleftarrow \text{NSym} \hookrightarrow & \text{MR} \\ | & | & | \\ \text{Sym} \hookrightarrow & \text{QSym} \twoheadleftarrow & \text{MR} \end{array}$$

Here Sym and NSym are Hopf subalgebras of QSym and MR respectively, while Sym and QSym are Hopf quotients of NSym and MR respectively. The vertical lines denote Hopf duality, so that Sym and MR are self-dual.

Each of the four Hopf algebras above come with a distinguished basis (see Section 2). In the case of the symmetric functions Sym this is the basis $\{s_\lambda\}$ of Schur functions. It is well known that (besides many other manifestations) Schur functions represent the Schubert classes $[X_\lambda]$ in the cohomology $H^*(\mathrm{Gr}(k, \mathbb{C}^n))$ of the Grassmannians $\mathrm{Gr}(k, \mathbb{C}^n)$ of k-planes in $\mathbb{C}^n$.

In [12] Lascoux and Schützenberger introduced the *Grothendieck polynomials* as representatives of K-theory classes of structure sheaves of Schubert varieties. Fomin and Kirillov in [5] studied these from combinatorial point of view. In particular, they introduced the *stable Grothendieck polynomials G_λ*, which are symmetric power series obtained as a limit of Grothendieck polynomials. In [3], Buch gave a combinatorial expression for stable Grothendieck polynomials as generating series of *set-valued tableaux*. These symmetric functions G_λ play the role of Schur functions in the K-theory $K^\circ(\mathrm{Gr}(k, n))$ of Grassmannians; roughly speaking G_λ represents the class of the structure sheaf of a Schubert variety. Buch studies a bialgebra Γ

spanned by the stable Grothendieck polynomials. Taking the completion of the bialgebra Γ, one can define a Hopf algebra which we denote $\mathfrak{m}\text{Sym}$.

Our investigation began with the observation that Buch's definition of set-valued tableaux can be extended to a definition of set-valued P-partitions, thus allowing one to define a "K-theoretic" analogue $\mathfrak{m}\text{QSym}$ of the Hopf algebra QSym of quasisymmetric functions. In fact the the entire diagram (1) can be extended to give the following diagram:

$$
\begin{array}{ccccc}
\mathfrak{M}\text{Sym} & \twoheadleftarrow & \mathfrak{M}\text{NSym} & \hookrightarrow & \mathfrak{M}\text{MR} \\
\big| & & \big| & & \big| \\
\mathfrak{m}\text{Sym} & \hookrightarrow & \mathfrak{m}\text{QSym} & \twoheadleftarrow & \mathfrak{m}\text{MR}
\end{array}
$$

Here $\mathfrak{M}\text{Sym}$ and $\mathfrak{m}\text{QSym}$ are Hopf quotients of $\mathfrak{M}\text{NSym}$ and $\mathfrak{m}\text{MR}$ respectively, while $\mathfrak{m}\text{Sym}$ and $\mathfrak{M}\text{NSym}$ are Hopf subalgebras of $\mathfrak{m}\text{QSym}$ and $\mathfrak{M}\text{MR}$ respectively. The vertical lines denote Hopf duality. Note that $\mathfrak{m}\text{Sym}$ and $\mathfrak{m}\text{MR}$ are no longer self-dual as combinatorial Hopf algebras.

Before describing our results in more detail we make some general remarks by grouping the six Hopf algebras into two groups: the $\mathfrak{m}$-*world* consisting of $\mathfrak{m}\text{Sym}$, $\mathfrak{m}\text{QSym}$ and $\mathfrak{m}\text{MR}$, and the $\mathfrak{M}$-*world* consisting of $\mathfrak{M}\text{Sym}$, $\mathfrak{M}\text{NSym}$ and $\mathfrak{M}\text{MR}$. The stable Grothendieck polynomials G_λ are in some sense deformations of the Schur functions s_λ; in particular, the lowest degree component of G_λ is equal to s_λ. In the same spirit, we observe in the $\mathfrak{m}$-world that

(1) the classical basis constitute the *lowest* degree components of the new basis;
(2) the product in the distinguished basis is infinite (with the exception of $\mathfrak{m}\text{Sym}$); both the product and coproduct consist of classical terms plus terms of *higher* degree.

In the $\mathfrak{M}$-world we have:

(1) the classical basis constitute the *highest* degree components of the new basis;
(2) the product and coproduct are finite and consist of classical terms plus terms of *lower* degree.

Besides the study of the Hopf structure of these six Hopf algebras, our main results also include: in the context of $\mathfrak{m}\text{QSym}$, a theory of set-valued P-partitions; and in the context of $\mathfrak{M}\text{Sym}$ and $\mathfrak{m}\text{Sym}$ the study of three new families of symmetric functions. These symmetric functions are weight generating functions of *weak set-valued tableaux*, *valued-set tableaux* and *reverse plane partitions*.

We now describe the structure and results of this paper in more detail. In Section 2 we review some standard results concerning the four (by now classical) Hopf algebras of (1).

1.1. $\mathfrak{m}\text{MR}$. As a preliminary step, we introduce in Section 3 the *multi-shuffle bialgebra* which comes with *multi-shuffle product* and *cuut coproduct*, but leave open the question of whether an antipode can be defined. In Section 4 we describe the *small multi-Malvenuto-Reutenauer Hopf algebra* $\mathfrak{m}\text{MR}$ in terms of a basis $w \in S_\infty^{\mathfrak{m}}$ of $\mathfrak{m}$-*permutations*. The fact that $\mathfrak{m}\text{MR}$ has an antipode is delayed till Section 7.4.

1.2. $\mathfrak{m}\text{QSym}$. In Section 5 we describe the Hopf algebra $\mathfrak{m}\text{QSym}$ of *multi quasisymmetric functions*. We first define $\mathfrak{m}\text{QSym}$ as a Hopf quotient of $\mathfrak{m}\text{MR}$. Next, for

a labeled poset (P, θ) we define *set-valued P-partitions* and show that the generating function $\tilde{K}_{P,\theta}$ is a quasisymmetric function which expands as a sum of *multi-fundamental quasisymmetric functions* $\tilde{L}_\alpha$ over a *multi-Jordan-Holder set* $\mathcal{J}(P, \theta)$. We show that the Hopf-algebra of formal linear combinations of the $\tilde{L}_\alpha$ (as α varies over all compositions) is isomorphic to $\mathfrak{m}$QSym. In addition we study the transition matrix between $\{\tilde{L}_\alpha\}$ and the (classical) fundamental quasisymmetric functions $\{L_\alpha\}$.

1.3. $\mathfrak{m}$Sym. In Section 6 we recall precisely the relationship ([3]) between $\mathfrak{m}$Sym and the K-theory of Grassmannians. We briefly discuss the Fomin-Greene method ([4]) for obtaining stable Grothendieck polynomials from operators which act on the space of partitions.

1.4. $\mathfrak{M}$MR. In Section 7 we enter the big $\mathfrak{M}$-world by describing the *big Multi-Malvenuto-Reutenauer Hopf algebra* $\mathfrak{M}$MR in terms of a basis $w \in S_\infty^{\mathfrak{M}}$ of $\mathfrak{M}$-*permutations*. We show that $\mathfrak{M}$MR is dual to $\mathfrak{m}$MR and describe an intriguing partial order on $\mathfrak{M}$-permutations, generalizing the usual weak order of the symmetric group.

1.5. $\mathfrak{M}$NSym. In Section 8 we describe and study the Hopf subalgebra $\mathfrak{M}$NSym $\subset$ $\mathfrak{M}$MR of *Multi-noncommutative symmetric functions* in the basis $\{\tilde{R}_\alpha\}$ which are analogues of (noncommutative) ribbon Schur functions. We show that $\mathfrak{m}$QSym and $\mathfrak{M}$NSym are Hopf-dual.

1.6. $\mathfrak{M}$Sym. In Section 9 we describe the Hopf algebra $\mathfrak{M}$Sym of *Multi-symmetric functions*. As an abstract Hopf algebra $\mathfrak{M}$Sym is isomorphic to Sym, but $\mathfrak{M}$Sym is equipped with a distinguished basis $\{g_\lambda\}$ of *dual stable Grothedieck polynomials* which are weight generating functions of reverse plane partitions. We show that the g_λ's are symmetric and Schur positive, and describe an explicit rule for decomposing them into basis of Schur functions. We show that $\mathfrak{M}$Sym and $\mathfrak{m}$Sym are Hopf-dual and that $\{g_\lambda\}$ and $\{G_\lambda\}$ are dual bases. We make explicit here the relation between $\mathfrak{M}$Sym and the K-homology of Grassmannian: the basis $\{g_\lambda\}$ represent the classes in K-homology of the ideal sheaves of the boundaries of Schubert varieties. In Sections 9.6-9.8 we introduce, again using the Fomin-Greene method, *weak set-valued tableaux* and *valued-set tableaux*. The weight generating functions of these tableaux describe the images of G_λ and g_λ under the involution $\omega : \text{Sym} \to \text{Sym}$ of the symmetric functions which sends the elementary symmetric functions e_n to the homogeneous symmetric functions h_n.

New analogues of the Loday-Ronco Hopf algebra of planar binary trees [11] can also be defined in the spirit of this paper. These Hopf algebras will be the subject of another article [1].

Remark 1.1. The reader mostly interested in the stable Grothendieck polynomials G_λ and dual stable Grothendieck polynomials g_λ can safely restrict his or her attention to Sections 5, 6 and 9.

Maria Ronco for drawing our attention to the results of [9] and [14]. We are grateful to Sergey Fomin for making useful suggestions concerning the presentation of the material. We would also like to thank Mark Shimozono and Mike Zabrocki for making their manuscript [15] available to us.

2. Four combinatorial Hopf algebras

In this section we briefly describe the four combinatorial Hopf algebras, MR, QSym, NSym, and Sym which we intend to generalize (see (1)).

We begin with some notation concerning compositions. A *composition* of n is a sequence $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ of positive integers such that $\alpha_1 + \alpha_2 + \cdots + \alpha_k = n$. We write $|\alpha| = n$. Denote the set of compositions of n by $\mathrm{Comp}(n)$. Associated to a composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ of n is a descent subset $D(\alpha) = \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \alpha_2 + \cdots + \alpha_{k-1}\}$ of $[n-1] = \{1, 2, \ldots, n-1\}$. The map $\alpha \mapsto D(\alpha)$ is a bijection between compositions of n and subsets of $[n-1]$. We will denote the inverse map by $\mathcal{C} : 2^{[n-1]} \to \mathrm{Comp}(n)$ so that $\mathcal{C}(D(\alpha)) = \alpha$.

Now if $w \in S_n$ is a permutation we let $\mathrm{Des}(w) = \{i \in [n-1] \mid w_i > w_{i+1}\}$ denote its descent set, and define $\mathcal{C}(w) = \mathcal{C}(\mathrm{Des}(w))$. If $\alpha \in \mathrm{Comp}(n)$, we let $w(\alpha)$ denote any permutation such that $\mathcal{C}(w(\alpha)) = \alpha$. Similarly, we define $w(D)$ for a subset D of $[n-1]$. Note that when we compare two descent sets, for example $\mathrm{Des}(w)$ and $D(\alpha)$, we always compare them as subsets. Thus $\mathrm{Des}(w)$ and $D(\alpha)$ will never be equal unless $w \in S_{|\alpha|}$.

A partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l > 0)$ is a decreasing composition. To a partition λ one may associate its Young diagram, also denoted λ, which (in English notation) is drawn as a set of boxes top-left justified, with λ_i boxes in the i-th row. If $\mu \subset \lambda$ then one obtains a skew Young diagram λ/μ by taking the set-theoretic difference of λ and μ. We let Λ denote the set of all partitions.

2.1. **The Malvenuto-Reutenauer Hopf algebra of permutations.** Malvenuto and Reutenauer [13] have defined a Hopf algebra structure MR on the free $\mathbb{Z}$-module spanned by the set of all permutations $S_\infty = \cup_n S_n$ (by convention $S_0 = \{\emptyset\}$ contains the empty permutation). For two permutations $w \in S_n$ and $v \in S_m$ define the *shuffle product* $w \cdot v$ by

$$w \cdot v = w_1 \ldots w_n \times (v_1 + n) \ldots (v_m + n),$$

where $\times$ denotes the usual shuffle of words: for example, $ab \times cd = abcd + acbd + acdb + cadb + cdab + cabd$. Now define the *cut coproduct* Δ by

$$\Delta(w) = \sum_{[u,v]=w} \mathrm{st}(u) \otimes \mathrm{st}(v),$$

where $[u, v]$ denotes the concatenation of u and v and $\mathrm{st}(.)$ is the standardization operator that replaces any sequence of distinct integers to the unique permutation with the same set of inversions. The unit map $\eta : \mathbb{Z} \to \mathrm{MR}$ is given by $\eta(1) = \emptyset$ while the counit $\varepsilon : \mathrm{MR} \to \mathbb{Z}$ extracts the coefficient of $\emptyset$. With this data $(\cdot, \Delta, \eta, \emptyset)$, the space MR is a bialgebra. In addition, it has an antipode (see [2]) which endows it with the structure of a Hopf algebra.

The Hopf algebra MR is self dual under the map $w \mapsto (w^{-1})^*$ where $\{w^*\}$ is the basis of MR^* dual to $\{w\}$.

2.2. The Hopf algebra of quasisymmetric functions.

A formal power series $f = f(x) \in \mathbb{Z}[[x_1, x_2, \ldots]]$ with bounded degree is called *quasisymmetric* if for any $a_1, a_2, \ldots, a_k \in \mathbb{P}$ we have

$$\left[x_{i_1}^{a_1} \cdots x_{i_k}^{a_k}\right] f = \left[x_{j_1}^{a_1} \cdots x_{j_k}^{a_k}\right] f$$

whenever $i_1 < \cdots < i_k$ and $j_1 < \cdots < j_k$. Here $[x_{i_1}^{a_1} \cdots x_{i_k}^{a_k}]f$ denotes the coefficient of $x_{i_1}^{a_1} \cdots x_{i_k}^{a_k}$ in f. Denote by $\mathrm{QSym} \subset \mathbb{Z}[[x_1, x_2, \ldots]]$ the ring of quasisymmetric functions.

The ring QSym has a natural coproduct which can be obtained as follows. Let $y_1, y_2, \ldots$ be another set of variables and order $\{x_i \cup y_j\}$ by $x_1 < x_2 < \cdots < y_1 < y_2 < \cdots$. With this order each $f \in \mathrm{QSym}$ determines $f(x, y) \in \mathbb{Z}[[x_1, x_2, \ldots, y_1, y_2, \ldots]]$. One may rewrite $f(x, y)$ as an element of $\mathbb{Z}[[x_1, x_2, \ldots]] \otimes \mathbb{Z}[[y_1, y_2, \ldots]]$ to obtain a coproduct map $\Delta : \mathrm{QSym} \to \mathrm{QSym} \otimes \mathrm{QSym}$. With this coproduct, QSym becomes a graded, connected, commutative but not cocommutative Hopf algebra. We omit the explicit formula for the antipode.

There are two distinguished $\mathbb{Z}$-bases of QSym, both labeled by compositions. Let α be a composition of n. The *monomial quasisymmetric function* M_α is given by

$$M_\alpha = \sum_{i_1 < \cdots < i_k} x_{i_1}^{\alpha_k} \cdots x_{i_k}^{\alpha_k}.$$

The *fundamental quasisymmetric function* L_α is given by

$$L_\alpha = \sum_{D(\beta) \supset D(\alpha)} M_\beta = \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_n \\ k \in D(\alpha) \Rightarrow i_{k+1} > i_k}} x_{i_1} x_{i_2} \cdots x_{i_n},$$

where the first summation is over compositions β satisfying $|\beta| = |\alpha|$.

The surjective map $w \mapsto L_{\mathcal{C}(w)}$ is a Hopf morphism (see [13]) which exhibits QSym as a quotient of MR.

2.3. The Hopf algebra of noncommutative symmetric functions.

We refer to [19] for more details concerning the material of this section. Let NSym denote the subspace of MR spanned by the elements

$$R_\alpha = \sum_{\mathrm{Des}(w^{-1}) = D(\alpha)} w$$

for each composition α. It turns out that NSym is a graded, connected, cocommutative but not commutative Hopf subalgebra of MR. In the basis $\{R_\alpha\}$, the multiplication can be written as

$$R_\alpha \, R_\beta = R_{\alpha \triangleright \beta} + R_{\alpha \triangleleft \beta},$$

where for $\alpha = (\alpha_1, \ldots, \alpha_k)$ and $\beta = (\beta_1, \ldots, \beta_l)$ we have $\alpha \triangleright \beta = (\alpha_1, \ldots, \alpha_k + \beta_1, \ldots, \beta_l)$ and $\alpha \triangleleft \beta = (\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_l)$.

Remark 2.1. Note that in [19] the notation $\alpha \cdot \beta$ is used for what we call here $\alpha \triangleleft \beta$. We however change the notation so that $\alpha \cdot \beta$ is saved for a different operation to be defined later.

In fact NSym is a free (noncommutative) algebra generated by symbols $\{S_i \mid i \geq 1\}$. Given a composition α we let $S_\alpha = S_{\alpha_1} S_{\alpha_2} \cdots S_{\alpha_k}$. Then

$$S_\alpha = \sum_{D(\beta) \subset D(\alpha)} R_\beta.$$

In terms of the generators $\{S_i \mid i \geq 1\}$, we have $\Delta S_i = \sum_{k=0}^{i} S_k \otimes S_{i-k}$.

The two Hopf algebras NSym and QSym are Hopf dual with $\{R_\alpha\}$ and $\{L_\alpha\}$ forming dual bases.

2.4. The Hopf algebra of (commutative) symmetric functions.

A formal power series $f = f(x) \in \mathbb{Z}[[x_1, x_2, \ldots]]$ with bounded degree is called *symmetric* if for any $a_1, a_2, \ldots, a_k \in \mathbb{P}$ we have

$$\left[x_{i_1}^{a_1} \cdots x_{i_k}^{a_k} \right] f = \left[x_{j_1}^{a_1} \cdots x_{j_k}^{a_k} \right] f$$

whenever $i_1, \ldots, i_k$ are all distinct and $j_1, \ldots, j_k$ are all distinct. Denote by Sym $\subset$ $\mathbb{Z}[[x_1, x_2, \ldots]]$ the algebra of symmetric functions. Every symmetric function is quasisymmetric and in fact Sym is a commutative and cocommutative Hopf subalgebra of QSym.

The Hopf algebra of symmetric functions has a distinguished basis $\{s_\lambda \mid \lambda \in \Lambda\}$ of Schur functions indexed by the set of all partitions. The Schur function s_λ is the weight generating function of semistandard tableaux with shape λ. The *Hall inner product* of Sym is defined by $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$. With this inner product, Sym is a self-dual Hopf algebra. The product and coproduct structure constants of Sym are both given by the *Littlewood-Richardson coefficients* $c_{\lambda\mu}^\nu \in \mathbb{Z}$:

$$s_\lambda s_\mu = \sum_\nu c_{\lambda\mu}^\nu s_n u \qquad \Delta(s_\lambda) = \sum_{\mu,\nu} c_{\mu\nu}^\lambda s_\mu \otimes s_\nu.$$

A skew shape λ/μ is a ribbon if it is connected and contains no 2×2 square. Given a composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$, there is a ribbon $r_\alpha = \lambda/\mu$ with α_k boxes in its first row, α_{k-1} boxes in its second row and so on. The map $R_\alpha \mapsto s_{r_\alpha}$ expresses Sym as the commutative quotient of NSym.

3. MULTI-SHUFFLE ALGEBRA

In the Sections 4-6 we will define and study three Hopf algebras: the small multi-Malvenuto-Reutenauer algebra $\mathfrak{m}$MR, the algebra of multi-quasisymmetric functions $\mathfrak{m}$QSym and the algebra of multi-symmetric functions $\mathfrak{m}$Sym. The Hopf algebra $\mathfrak{m}$QSym is a Hopf quotient of $\mathfrak{m}$MR while $\mathfrak{m}$Sym is a Hopf subalgebra of $\mathfrak{m}$QSym. Now, as a preliminary step, we define the multi-shuffle algebra.

In the following we will be dealing with free $\mathbb{Z}$-modules M which are the sets of *arbitrary* $\mathbb{Z}$-linear combinations of a countable set S. If N is another $\mathbb{Z}$-module, we will say that a linear function $M \to N$ is *continuous* if it respects arbitrary linear combinations of elements of S (not just finite linear combinations).

Let $\mathfrak{A}$ denote an alphabet, $\mathfrak{A}^*$ denote the set of (possibly empty) words with letters from $\mathfrak{A}$ and let $\mathfrak{m}S[\mathfrak{A}^*] = \prod_{a \in \mathfrak{A}^*} \mathbb{Z}a$ denote the $\mathbb{Z}$-module of (infinite) $\mathbb{Z}$-linear combinations words from $\mathfrak{A}^*$.

Let $u = u_1 u_2 \cdots u_k \in \mathfrak{A}^*$ be a word. Call $w = w_1 w_2 \cdots w_m$ a *multiword* of u if there is a surjective and non-decreasing map $t : [m] \longrightarrow [k]$, so that $w_j = u_{t(j)}$. Let $u = u_1 u_2 \cdots u_k$ and $v = v_1 v_2 \cdots v_l$ be two words, and assume that all letters v_i and u_j are distinct. Then a word $w = w_1 w_2 \cdots w_m$ is a *multishuffle* of u and v if

    (1) neighboring letters of w are distinct, that is for any i we have $w_i \neq w_{i+1}$;
    (2) when restricted to alphabets $\{v_i\}$ and $\{u_j\}$ word w becomes a multiword of v and u correspondingly.

We denote by $u \star v$ the sum in $\mathfrak{m}S[\mathfrak{A}^*]$ of all multishuffles of u and v. Now suppose that $x = x_1 x_2 \cdots x_k$ and $y = y_1 y_2 \cdots y_l$ are two other words, with possibly repeated letters. Then $x \star y$ is obtained from $u \star v$ by changing each u_i to x_i and each v_j to y_j. It is possible to attain multiplicities in this way. For example, we have

$$ab \star a = aba + 2aab + 2aaab + 2aaba + abab + \ldots.$$

We extend $\star$ to $\mathfrak{m}S[\mathfrak{A}^*]$ by linearity and continuity (it is a quick check to verify that this extension is well-defined). One can give the following recursive definition of multishuffle product.

Proposition 3.1. *Let $v = v_1 v'$ and $u = u_1 u'$ where v_1, u_1 are letters and v', u' are words. Then*

$$v \star u = (v_1 + u_1 v_1 + v_1 u_1 v_1 + \ldots)(v' \star u) + (u_1 + v_1 u_1 + u_1 v_1 u_1 + \ldots)(v \star u').$$

We shall also use the same notation for the multiplication map $\star : \mathfrak{m}S[\mathfrak{A}^*] \otimes \mathfrak{m}S[\mathfrak{A}^*] \to \mathfrak{m}S[\mathfrak{A}^*]$. Multishuffling is commutative and associative:

Lemma 3.2. *For any three words $u, v, x \in \mathfrak{A}^*$, we have $u \star v = v \star u$ and $(u \star v) \star x = u \star (v \star x)$.*

Proof. The first statement is immediate from the definition. For the second statement, first assume that all the letters in u, v, x are distinct. Then both $(u \star v) \star x$ and $u \star (v \star x)$ are equal to the set of words w satisfying (a) for any i, we have $w_i \neq w_{i+1}$, and (b) when restricted to alphabets $\{v_i\}$, $\{u_j\}$ and $\{x_k\}$, the word w becomes a multiword of v, u and x correspondingly. The general case also follows immediately. $\square$

We now define the *cuut* coproduct structure on $\mathfrak{m}S[\mathfrak{A}^*]$, in analogy with the cut coproduct. Let $a = a_1 a_2 a_3 \ldots a_n \in \mathfrak{A}^*$. Then define

$$\blacktriangle(a) = \emptyset \otimes a_1 a_2 a_3 \ldots a_n + a_1 \otimes a_1 a_2 a_3 \ldots a_n + a_1 \otimes a_2 a_3 \ldots a_n + a_1 a_2 \otimes a_2 a_3 \ldots a_n +$$

$$a_1 a_2 \otimes a_3 \ldots a_n + \cdots + a_1 a_2 a_3 \ldots a_n \otimes a_n + a_1 a_2 a_3 \ldots a_n \otimes \emptyset.$$

When a letter a_i occurs twice in a term in the above expression we say that a_i has been "cut in the middle" to obtain such a term. For example, we have

$$\blacktriangle(cut) = \emptyset \otimes cut + c \otimes cut + c \otimes ut + cu \otimes ut + cu \otimes t + cut \otimes t + cut \otimes \emptyset.$$

We extend $\blacktriangle$ to $\mathfrak{m}S[\mathfrak{A}^*]$ by linearity and continuity. Define the *unit* map $\eta : \mathbb{Z} \to \mathfrak{m}S[\mathfrak{A}^*]$ by $\eta(n) = n.\emptyset$ and the counit map $\varepsilon : \mathfrak{m}S[\mathfrak{A}^*] \to \mathbb{Z}$ by letting ε take the coefficient of $\emptyset$.

Theorem 3.3. *The space $\mathfrak{m}S[\mathfrak{A}^*]$ forms a bialgebra with multi-shuffle product $\star$, cuut coproduct $\blacktriangle$, unit η and counit ε.*

We call $\mathfrak{m}S[\mathfrak{A}^*]$ the *multi-shuffle algebra*.

Proof. It is easy to verify that $\mathfrak{m}S[\mathfrak{A}^*]$ is both a unital associative algebra and a counital coassociative coalgebra. We now verify the compatibility of $\star$ and $\blacktriangle$. Let w and u be two words which we assume for simplicity to have distinct letters. Then $\blacktriangle(w \star u)$ is a linear combination of all terms $x \otimes y$ such that either (1) x is a term in $w' \star u'$ and y is a term in $w'' \star u''$, where $w = w'w''$ and $u = u'u''$, or (2) x is a term in $w'a \star u'$ and y is a term in $aw'' \star u''$ where $w = w'aw''$ and $u = u'u''$, or (3) x is a term in $w' \star u'b$ and y is a term in $w'' \star bu''$ where $w = w'w''$ and $u = u'bu''$,

or (4) x is a term in $w'a \star u'b$ and y is a term in $aw'' \star bu''$ where $w = w'aw''$ and $u = u'bu''$. Here a, b are letters while w', w'', u', u'' are words. For example, case (2) or (4) occurs if some letter a in w lies to both sides of the cutting point or is cut in the middle when we apply $\blacktriangle$; otherwise case (1) or (3) occurs. We check that the same four kind of terms occur in $\blacktriangle(w) \star \blacktriangle(u)$. □

Problem 3.4. Does $\mathfrak{m}S[\mathfrak{A}^*]$ have an antipode?

4. The small multi-Malvenuto-Reutenauer Hopf algebra

Definition 4.1. A *small multi-permutation*, or $\mathfrak{m}$-permutation of $[n]$ is a word w in the alphabet $1, \ldots, n$ such that no two consecutive letters in w are equal. The *length* $\ell(w)$ of w is the the number of letters in w.

We denote the set of multi-permutations of $[n]$ by $S_n^{\mathfrak{m}}$ and the set of all multi-permutations by $S_\infty^{\mathfrak{m}} = \cup_{n \geq 0} S_n^{\mathfrak{m}}$. By convention $S_0^{\mathfrak{m}}$ contains a single element – the empty multi-permutation $\emptyset$. We let $\mathfrak{m}MR = \prod_{w \in S_\infty^{\mathfrak{m}}} \mathbb{Z}.w$ be the free $\mathbb{Z}$-module of arbitrary $\mathbb{Z}$-linear combinations of multi-permutations.

Let $w = w_1 \ldots w_k$ and $u = u_1 \ldots u_l$ be two multi-permutations, and assume that $w \in S_n^{\mathfrak{m}}$, $u \in S_m^{\mathfrak{m}}$. Define the product $w * u$ of w and u as follows:

$$w * u = w \star (u + n) = w_1 \ldots w_k \star (u_1 + n) \ldots (u_l + n)$$

where to use the shuffle product $\star$ we treat w and $u + n$ as words in the alphabet $\mathbb{N}$. We extend the formula by linearity and continuity to give a multiplication $* : \mathfrak{m}MR \otimes \mathfrak{m}MR \to \mathfrak{m}MR$.

Define the standardization operator $\mathrm{st} : \mathbb{N}^* \to S_\infty^{\mathfrak{m}}$ by sending a word w to the unique $u \in S_\infty^{\mathfrak{m}}$ of the same length (if it exists) such that $w_i \leq w_j$ if and only if $u_i \leq u_j$ for each $1 \leq i, j \leq \ell(w)$. (Recall that $\mathbb{N}^*$ denotes the set of words in the alphabet $\{1, 2, 3, \ldots\}$.) We define the coproduct $\Delta(w)$ by extending via linearity and continuity the cuut coproduct as follows:

$$\Delta(w) = \mathrm{st}(\blacktriangle w)$$

where we have extended by linearity and continuity the definition of st.

Define the unit map $\eta : \mathbb{Z} \to \mathfrak{m}MR$ by $\eta(n) = n \cdot \emptyset$ and the counit map $\varepsilon : \mathfrak{m}MR \to \mathbb{Z}$ by letting ε take the coefficient of $\emptyset$.

Theorem 4.2. *The space $\mathfrak{m}MR$ is a bialgebra with product $*$, coproduct Δ, unit η, and counit ε.*

Proof. That $*$ is associative and Δ is coassociative follows from the corresponding properties in the multi-shuffle algebra. The proof that $*$ and Δ are compatible is essentially the same as for Theorem 3.3. The only observation needed is that the standardization operator st can be applied at the very end of the calculation of $\Delta(w) * \Delta(u)$ instead of immediately after calculating $\blacktriangle(w)$ and $\blacktriangle(u)$. □

We shall call the bialgebra $\mathfrak{m}MR$ of Theorem 4.2 the *small multi-Malvenuto-Reutenauer bialgebra*. We shall show later that $\mathfrak{m}MR$ has an antipode, making it a Hopf algebra.

Remark 4.3. Call an element $w \in S_\infty^{\mathfrak{m}}$ *irreducible* if it cannot be written in the form $w = v/u = v_1 \ldots v_k(u_1 + n) \ldots (u_l + n)$ for two smaller $\mathfrak{m}$-permutations $v \in S_n^{\mathfrak{m}}$ and u. The following simple observations say that combinatorially $\mathfrak{m}MR$ is "free" over

the set of irreducible elements (this statement is difficult to make precise because $\mathfrak{m}$MR is a completion):

(1) Every $\mathfrak{m}$-permutation $w \in S_\infty^{\mathfrak{m}}$ can be uniquely written as $w^1/w^2/\cdots/w^k$ where the $w^i \in S_\infty^{\mathfrak{m}}$ are irreducible. We say that w is k-reducible in this case.
(2) If $w^1, w^2, \ldots, w^k$ are irreducible, the only term in $w^1 * w^2 * \cdots * w^k$ which is k-reducible is $w^1/w^2/\cdots/w^k$.

5. Set-valued P-partitions and multi-quasisymmetric functions

5.1. The Hopf-algebra $\mathfrak{m}$QSym.

For $w \in S_\infty^{\mathfrak{m}}$ we define the descent set $\mathrm{Des}(w) \subset [1, \ell(w) - 1]$ by

$$\mathrm{Des}(w) = \{i \in [1, \ell(w) - 1] \mid w_i > w_{i+1}\}.$$

Thus for example $\mathrm{Des}(15132342) = \{2, 4, 7\}$. Note that as in Section 2 by convention descent sets are always considered as subsets (of $[1, \ell(w) - 1]$) so the descent sets of $w, u \in S_\infty^{\mathfrak{m}}$ can only coincide if $\ell(w) = \ell(u)$. Let $I \subset \mathfrak{m}$MR denote the free $\mathbb{Z}$-submodule spanned by the elements $w - u$ for pairs w, u satisfying $\mathrm{Des}(w) = \mathrm{Des}(u)$.

Lemma 5.1. *The subspace I is a biideal of $\mathfrak{m}$MR. In other words, we have $I * \mathfrak{m}MR \subset I$, $\mathfrak{m}MR * I \subset I$, and $\Delta(I) \subset I \otimes \mathfrak{m}MR + \mathfrak{m}MR \otimes I$.*

Proof. A term $v \in S_\infty^{\mathfrak{m}}$ occurring in the product $w * u$ of $w \in S_n^{\mathfrak{m}}$ and $u \in S_m^{\mathfrak{m}}$ is determined by the following information: (1) the location in v of letters in $[1, n]$ (and hence also $[n+1, n+m]$), (2) the restriction of v to the alphabet $[1, n]$ (which is a multiword of w), and (3) the restriction of v to the alphabet $[n+1, n+m]$ (which is a multiword of u). If $w, w' \in S_\infty^{\mathfrak{m}}$ have the same length then there is a canonical bijection between the multiwords of w and of w'. Thus if $w, w', u, u' \in S_\infty^{\mathfrak{m}}$ are such that $\ell(w) = \ell(w')$ and $\ell(u) = \ell(u')$ then there is a canonical bijection Φ between the terms in $w * u$ and those in $w' * u'$. If in addition, w and w' have the same descent set and u and u' have the same descent set then Φ preserves descent sets. This proves that I is an (algebra) ideal.

Now suppose $w, u \in S_\infty^{\mathfrak{m}}$ satisfy $\mathrm{Des}(w) = \mathrm{Des}(u)$ and thus in particular $\ell(w) = \ell(u)$. There is a canonical bijection between the terms of $\Delta(w)$ and those of $\Delta(u)$ so that if $w' \otimes w''$ corresponds to $u' \otimes u''$ then $\mathrm{Des}(w') = \mathrm{Des}(u')$ and $\mathrm{Des}(w'') = \mathrm{Des}(u'')$. We show that $w' \otimes w'' - u' \otimes u'' \in I \otimes \mathfrak{m}MR + \mathfrak{m}MR \otimes I$. This follows from

$$w' \otimes w'' - u' \otimes u'' = w' \otimes (w'' - u'') + (w' - u') \otimes u''.$$

$\square$

By Lemma 5.1, the quotient space $\mathfrak{m}MR/I$ is a quotient Hopf algebra, which we call $\mathfrak{m}$QSym. We will give an explicit model for $\mathfrak{m}$QSym as an algebra of quasisymmetric functions. In [8], Hazewinkel shows that QSym is a free algebra over $\mathbb{Z}$. It would be interesting to investigate this for $\mathfrak{m}$QSym as well.

5.2. Posets and P-partitions.

We recall the basic definitions concerning P-partitions [17]. Let P be a finite poset with n elements and $\theta : P \to [n]$ be a bijective labeling of P.

Definition 5.2. A (P, θ)-*partition* is a map $\sigma : P \to \mathbb{P}$ such that for each covering relation $s \lessdot t$ in P we have

$$\sigma(s) \leq \sigma(t) \qquad\qquad \text{if } \theta(s) < \theta(t),$$
$$\sigma(s) < \sigma(t) \qquad\qquad \text{if } \theta(t) < \theta(s).$$

Denote by $\mathcal{A}(P, \theta)$ the set of all (P, θ)-partitions. If P is finite then one can define the formal power series $K_{P,\theta}(x_1, x_2, \ldots) \in \mathbb{Z}[[x_1, x_2, \ldots]]$ by

$$K_{P,\theta}(x_1, x_2, \ldots) = \sum_{\sigma \in \mathcal{A}(P,\theta)} x_1^{\#\sigma^{-1}(1)} x_2^{\#\sigma^{-1}(2)} \cdots .$$

The composition $\mathrm{wt}(\sigma) = (\#\sigma^{-1}(1), \#\sigma^{-1}(2), \ldots)$ is called the *weight* of σ.

Recall that a linear extension of P is a bijection $e : P \to \{1, 2, \ldots, n\}$ satisfying $e(x) < e(y)$ if $x < y$ in P. The Jordan-Holder set $\mathcal{J}(P, \theta)$ of (P, θ) is the set

$$\{\theta(e^{-1}(1))\theta(e^{-1}(1)) \cdots \theta(e^{-1}(n)) \mid e \text{ is a linear extension of } P\} \subset S_n.$$

Theorem 5.3 ([16]). *The generating function $K_{P,\theta}$ is quasisymmetric. We have* $K_{P,\theta} = \sum_{w \in \mathcal{J}(P,\theta)} L_{D(w)}$.

5.3. **Set-valued P-partitions.** In this section we define set-valued P-partitions. Let $\tilde{\mathbb{P}}$ be the set of all non-empty finite subsets of $\mathbb{P}$. For $a \in \tilde{\mathbb{P}}$ we define $\min(a)$ and $\max(a)$ to be the minimal and maximal elements of a. Suppose that $a, b \in \tilde{\mathbb{P}}$. Then we say a is less than b and write $a \leq b$ if and only if $\max(a) \leq \min(b)$; similarly we have $a < b$ if and only if $\max(a) < \min(b)$. Note that $a \leq a$ if and only if $|a| = 1$, that is a consists of just one element. Let (P, θ) as before be a poset with a bijective labeling.

Definition 5.4. A (P, θ)-*set-valued partition* is a map $\sigma : P \to \tilde{\mathbb{P}}$ such that for each covering relation $s \lessdot t$ in P we have

$$\sigma(s) \leq \sigma(t) \qquad\qquad \text{if } \theta(s) < \theta(t),$$
$$\sigma(s) < \sigma(t) \qquad\qquad \text{if } \theta(t) < \theta(s).$$

Denote by $\tilde{\mathcal{A}}(P, \theta)$ the set of all (P, θ)-set-valued partitions. For $\sigma \in \tilde{\mathcal{A}}(P, \theta)$ denote by $|\sigma| = \sum_{a \in P} |\sigma(a)|$ the total number of letters used in the set-valued partition. For each $i \in \mathbb{P}$ we define $\sigma^{-1}(i) = \{x \in P \mid i \in \sigma(x)\}$. As before, the composition $\mathrm{wt}(\sigma) = (\#\sigma^{-1}(1), \#\sigma^{-1}(2), \ldots)$ is called the *weight* of σ. Define the formal power series $\tilde{K}_{P,\theta}(x_1, x_2, \ldots) \in \mathbb{Z}[[x_1, x_2, \ldots]]$ by

$$\tilde{K}_{P,\theta}(x_1, x_2, \ldots) = \sum_{\sigma \in \tilde{\mathcal{A}}(P,\theta)} x_1^{\#\sigma^{-1}(1)} x_2^{\#\sigma^{-1}(2)} \cdots .$$

It is easy to see that $\tilde{K}_{P,\theta}$ is always a quasisymmetric function. In the following example which will be of major importance for us $\tilde{K}_{P,\theta}$ happens to be a symmetric function.

Example 5.5. Let $P = \lambda$ be the poset of squares in the Young diagram of a partition $\lambda = (\lambda_1, \ldots, \lambda_l)$. Let θ_s be the labeling of λ obtained from the bottom to top row-reading order; in other words the bottom row of λ is labeled $1, 2, \ldots, \lambda_l$, the next row is labeled $\lambda_l + 1, \ldots, \lambda_{l-1} + \lambda_l$ and so on. Then K_{λ, θ_s} is equal to the Schur function s_λ, while $\tilde{K}_{\lambda, \theta_s}$ is (nearly) equal to the *stable Grothendieck polynomial* G_λ studied in [3]. We will return to this example in Section 6.

Let $\alpha \vdash n$ be a composition and let C be a chain $c_1 < c_2 < \ldots < c_n$ with n elements and $w = w_1 w_2 \ldots w_n \in S_n$ a permutation of $\{1, 2, \ldots, n\}$ such that $\mathcal{C}(w) = \alpha$. Then (C, w) can be considered a labeled poset, where $w(c_i) = w_i$. Now define the *multi-fundamental quasisymmetric functions* by $\tilde{L}_\alpha = \tilde{K}_{C,w}$ (clearly $\tilde{K}_{C,w}$ depends only on α).

A *linear multi-extension* of P by $[N]$ is a map $e : P \to 2^{\{1,2,\ldots,N\}}$ for some $N \geq n = |P|$ satisfying

(1) $e(x) < e(y)$ if $x < y$ in P,
(2) each $i \in [N]$ is in $e(x)$ for exactly one $x \in P$, and
(3) none of the sets $e(x)$ contains both i and $i + 1$ for any i.

The *multi-Jordan-Holder set* $\tilde{\mathcal{J}}(P, \theta) = \cup_N \tilde{\mathcal{J}}_N(P, \theta)$ of (P, θ) is the union of sets

$$\tilde{\mathcal{J}}_N(P, \theta) = \{\theta(e^{-1}(1))\theta(e^{-1}(2))\cdots\theta(e^{-1}(N))\}$$

where in the above formula e varies over the set of linear multi-extensions of P by $[N]$. The set $\tilde{\mathcal{J}}_N(P, \theta)$ is a subset of the set $\{w \in S_n^{\mathfrak{m}} \mid \ell(w) = N\}$ of $\mathfrak{m}$-permutations of length N on n letters. In fact if P is the antichain with n-elements and θ is any labeling then we have $\mathcal{J}_N(P, \theta) = \{w \in S_n^{\mathfrak{m}} \mid \ell(w) = N\}$. The following result is the set-valued analogue of Theorem 5.3.

Theorem 5.6. *We have* $\tilde{K}_{P,\theta} = \sum_{N \geq n} \sum_{w \in \tilde{\mathcal{J}}_N(P,\theta)} \tilde{L}_{\mathcal{C}(w)}$.

Proof. We give an explicit weight-preserving bijection between $\tilde{\mathcal{A}}(P, \theta)$ and the set of pairs (w, σ') where $w \in \tilde{\mathcal{J}}_N(P, \theta)$ and $\sigma' \in \tilde{\mathcal{A}}(C, w)$ where $C = (c_1 < c_2 < \cdots < c_l)$ is a chain with $l = \ell(w)$ elements. Let $\sigma \in \tilde{\mathcal{A}}(P, \theta)$. For each i, identify $\sigma^{-1}(i)$ with a subset of $[n]$ via θ and let $w_\sigma^{(i)}$ denote the word of length $|\sigma^{-1}(i)|$ obtained by writing the elements of $\sigma^{-1}(i)$ in increasing order. Let w be the unique $\mathfrak{m}$-permutation such that $w_\sigma := w_\sigma^{(1)} w_\sigma^{(2)} \cdots$ is a multiword of w, and we let $t : \ell(w_\sigma) \to \ell(w)$ denote the associated function as in Section 3. Note that $\sigma^{(r)} = \emptyset$ for sufficiently large r, so that w_σ is a finite word in the alphabet $[n]$ (using all the letters of $[n]$). Now define $\sigma' \in \tilde{\mathcal{A}}(C, w)$ by

$$\sigma'(c_i) = \{r \in \mathbb{P} \mid w_\sigma^{(r)} \text{ contributes letters to } w_\sigma|_{t^{-1}(i)}\}$$

where $w_\sigma|_{t^{-1}(i)}$ is the set of letters in w_σ at the positions in the interval $t^{-1}(i)$. We claim that this defines a map $\alpha : \sigma \mapsto (w, \sigma')$ with the required properties.

First, w is the multiword associated to the linear multi-extension e_w of P by $\ell(w)$ defined by the condition that $e_w(x)$ contains j if and only if $w_j = \theta(x)$. It follows from the definition that this $e_w : P \to 2^{[1, \ell(w)]}$ is a linear multi-extension. To check that σ' is a set-valued (C, w) partition, we note that $\sigma'(c_i) \leq \sigma'(c_{i+1})$ by definition, since the function t is non-decreasing. Furthermore, if $w_i > w_{i+1}$ then $\sigma'(c_i) < \sigma'(c_{i+1})$ because each $w_\sigma^{(r)}$ was defined to be increasing.

Finally, the inverse map $\beta : (w, \sigma') \mapsto \sigma$ can be defined by the formula

$$\sigma(x) = \bigcup_{j \in e_w(x)} \sigma'(c_j).$$

That the above union is disjoint follows from the fact that there is always a descent somewhere between two occurrences of the same letter in w. That σ as defined respects θ is due to the fact that e_w is a linear multi-extension. The equation $\beta \circ \alpha = \text{id}$ follows immediately. For $\alpha \circ \beta = \text{id}$, consider a subset $\sigma'(c_j) \subset \sigma(x)$.

One checks that this subset gives rise to $|\sigma'(c_j)|$ consecutive letters all equal to $\theta(x)$ in w_σ and that this is a maximal set of consecutive repeated letters. This shows that one can recover σ'. To see that w is recovered correctly, one notes that if $\sigma'(c_j)$ and $\sigma'(c_{j+1})$ contain the same letter r then $w_j < w_{j+1}$ so by definition w_j is placed correctly before w_{j+1} in $w_\sigma^{(r)}$.

$\square$

Example 5.7. We illustrate the proof of Theorem 5.6. Let θ_s be the labeling

3	4	5
1	2	

of the shape $\lambda = (3, 2)$ as in Example 5.5. Take the (λ, θ_s)-partition

12	235	5678
45	8	

in $\tilde{\mathcal{A}}(\lambda, \theta_s)$. Then we have

$$w_\sigma = (3; 3, 4; 4; 1; 1, 4, 5; 5; 5; 2, 5),$$

where for example $w_\sigma^{(2)} = (3, 4)$ since the cells labeled 3 and 4 contain the number 2 in σ. Therefore

$$w = (3, 4, 1, 4, 5, 2, 5)$$

and the corresponding composition $\mathcal{C}(w)$ is $(2, 3, 2)$. Then σ' if written as sequence is

$$\{1, 2\}, \{2, 3\}, \{4, 5\}, \{5\}, \{5, 6, 7\}, \{8\}, \{8\}.$$

For example $\sigma'(c_1) = \{1, 2\}$ since $w_\sigma^{(1)}$ and $w_\sigma^{(2)}$ contribute 3's into the beginning of w_σ.

The inverse map β can now be understood as follows: we parse w and σ' in parallel and place $\sigma'(c_i)$ into the cell $\theta_s^{-1}(w_i)$. For example we place $\{1, 2\}$ into the cell labeled 3, $\{2, 3\}$ into the cell labeled 4, and so on.

Example 5.8. We give an example of the decomposition of $\tilde{K}_{\lambda, \theta_s}$ into multi-fundamental quasisymmetric functions. Take $\lambda = (3, 1)$ and take the labeling θ_s of the cells as described in Example 5.5:

2	3	4
1		

Then the usual Jordan-Holder set consists of the sequences $(2, 1, 3, 4)$, $(2, 3, 1, 4)$, and $(2, 3, 4, 1)$. This coincides with the part $\tilde{\mathcal{J}}_4(\lambda, \theta_s)$ of the multi-Jordan-Holder set. The set $\tilde{\mathcal{J}}_5(\lambda, \theta_s)$ consists of the words $(2, 1, 3, 1, 4)$, $(2, 1, 3, 4, 1)$, $(2, 3, 1, 3, 4)$, $(2, 3, 1, 4, 1)$, and $(2, 3, 4, 1, 4)$. This gives us the following part of the decomposition

$$\tilde{K}_{(3,1),\theta_s} = \tilde{L}_{(1,3)} + \tilde{L}_{(2,2)} + \tilde{L}_{(3,1)} + \tilde{L}_{(1,2,2)} + \tilde{L}_{(1,3,1)} + \tilde{L}_{(2,3)} + \tilde{L}_{(2,2,1)} + \tilde{L}_{(3,2)} + \cdots$$

5.4. **Structure of $\mathfrak{m}$QSym.** Let α be a composition of n. We let $w(\alpha)$ denote any permutation such that $\alpha = \mathcal{C}(w(\alpha)) := \mathcal{C}(\text{Des}(w(\alpha)))$.

Proposition 5.9. *Let α be a composition of n and β be a composition of m. Then*

$$\tilde{L}_\alpha \tilde{L}_\beta = \sum_{u \in \text{Sh}^{\mathfrak{m}}(w(\alpha), w(\beta))} \tilde{L}_{\mathcal{C}(u)},$$

where $\text{Sh}^{\mathfrak{m}}(w(\alpha), w(\beta))$ denotes the set of multishuffles of $w(\alpha)$ and $w(\beta) + n :=$ $(w_1(\beta) + n)(w_2(\beta) + n) \cdots (w_m(\beta) + n)$.

Proof. Take two chains $C = c_1 < c_2 < \ldots < c_n$ and $C' = c'_1 < c'_2 < \ldots < c'_m$. We label the disjoint union poset $C \cup C'$ by setting $\theta(c_i) = w_i(\alpha)$ and $\theta(c'_i) = w_i(\beta) + n$. Then the multi-Jordan-Holder set $\tilde{\mathcal{J}}(C \cup C', \theta)$ is exactly the set of multishuffles of $w(\alpha)$ and $w(\beta) + n$. Since $\tilde{K}_{C,w(\alpha)} \tilde{K}_{C',w(\beta)} = \tilde{K}_{C \cup C', \theta}$ we obtain the claimed result by Theorem 5.6. $\square$

If $f(x)$ is a formal linear combination of the multi-fundamental quasisymmetric functions $\tilde{L}_\alpha(x)$, we let $f(x, y)$ denote the corresponding formal power series in the variables $x_1, x_2, \ldots, y_1, y_2, \ldots$ obtained by considering $f(x)$ as a quasisymmetric function (of unbounded degree).

Proposition 5.10. *Let α be a composition. Then*

$$\tilde{L}_\alpha(x, y) = \sum_{u \otimes u' \in \text{Cuut}^{\mathfrak{m}}(w(\alpha))} \tilde{L}_{\mathcal{C}(u)}(x) \otimes \tilde{L}_{\mathcal{C}(u')}(y)$$

where $\text{Cuut}^{\mathfrak{m}}(w(\alpha))$ denote the set of terms in the cuut coproduct of $w(\alpha)$.

Proof. The power series $\tilde{L}_\alpha(x, y)$ is the weight generating function of set-valued P-partitions of a labeled chain $(C, w(\alpha))$ using the ordered set of letters $1 < 2 < \cdots < 1' < 2' < \cdots$ where the unprimed letters i are given weight x_i and the primed letters i' are given weight y_i. There are two kinds of such set-valued P-partitions θ: either (1) there is some $k \in [1, |\alpha|]$ so that $\theta(c_i)$ contains only unprimed letters for $i \leq k$ and $\theta(c_j)$ contains only primed letters for $j > k$, or (2) there is a unique $k \in [1, |\alpha|]$ so that $\theta(c_k)$ uses both primed and unprimed letters. This gives rise to the two kinds of terms in the cuut coproduct: (1) corresponds to the terms which occur in the usual cut coproduct while (2) corresponds to the extra terms obtained by cutting in the middle of $w(\alpha)_k$. $\square$

Theorem 5.11. *The map $\psi : \mathfrak{m}\text{MR} \to \prod_\alpha \mathbb{Z}\tilde{L}_\alpha$ given by $w \mapsto L_{\mathcal{C}(w)}$ is a bialgebra morphism, identifying $\prod_\alpha \mathbb{Z}\tilde{L}_\alpha$ with $\mathfrak{m}$QSym.*

Proof. It is clear that the kernel of ψ is the ideal I of Lemma 5.1. The fact that ψ is a bialgebra morphism follows from Propositions 5.9 and 5.10 and the proof of Lemma 5.1. $\square$

5.5. **Further properties of the multi-fundamental quasisymmetric functions.** We begin by describing how to express $\tilde{L}_\alpha$ as a (infinite) linear combination of L_α's. For a formal power series $f(x_1, x_2, \ldots)$ of possibly unbounded degree we let $H_i(f)$ denote the homogeneous component of degree i. We define a family of linear maps $^{(i)} : \text{QSym} \to \text{QSym}$ by $L_\alpha^{(i)} = H_{i+|\alpha|}(\tilde{L}_\alpha)$, and extending by linearity. In particular $f^{(0)} = f$ for any $f \in \text{QSym}$.

Suppose $D \subset [n-1]$ is a subset thought of as a descent set and $E \subset [n+i-1]$. An injective and order-preserving map $t : [n-1] \to [n+i-1]$ is an i-*extension* of D to E if $t(D) \subset E$ and $(E \backslash t(D)) = ([n+i-1] \backslash t([n-1]))$. In other words, E is the union of $t(D)$ and the set of elements not in the image of t (in particular E is determined by t and D). An immediate consequence is that $|E| = |D| + i$. Note that there may be many i-extensions even when D and E are fixed. For example, if $D = [2] \subset [2]$ and $E = [3] \subset [3]$ then there are three 1-extensions of D to E, corresponding to the three injective order-preserving maps $t : [2] \to [3]$. We denote the set of i-extensions of D to E by $T(D, E)$.

Theorem 5.12. *Let α be a composition n and $D = D(\alpha)$ the corresponding descent set. Then for each $i \geq 0$, we have*

$$(2) \qquad L_\alpha^{(i)} = \sum_{E \subset [n+i-1]} |T(D,E)| \, L_{\mathcal{C}(E)}$$

and

$$(3) \qquad M_\alpha^{(i)} = \sum_{E \subset [n+i-1]} |T(D,E)| \, M_{\mathcal{C}(E)}.$$

For each $i, j \geq 0$ and $f \in \mathrm{QSym}$, one has $(f^{(i)})^{(j)} = \binom{i+j}{i} f^{(i+j)}$.

Proof. Let $w = w(\alpha)$ and consider the subset $\tilde{\mathcal{A}}_i(C, w) \subset \tilde{\mathcal{A}}(C, w)$ consisting of set-valued (C, w)-partitions σ of size $|\sigma| = n+i$. We must show that the generating function of $\tilde{\mathcal{A}}_i(C, w)$ is equal to $\sum_{E \subset [n+i-1]} |T(D,E)| \, L_{\mathcal{C}(E)}$. Indeed for each pair $t \in T(D, E)$ for some E, the function $L_{\mathcal{C}(E)}$ is the generating function of all $\sigma \in \tilde{\mathcal{A}}_i(C, w)$ satisfying $|\sigma(c_i)| = t(i) - t(i-1)$ where one defines $t(0) = 0$ and $t(n) = n+i$. Indeed one obtains a (usual) $(C', w(\mathcal{C}(E)))$-partition $\sigma' \in \mathcal{A}(C', w(E))$ by assigning the elements of $\sigma(c_i)$ in increasing order to $c'_{t(i-1)+1}, \ldots, c'_{t(i)}$, where $C' = c'_1 < c'_2 < \ldots < c'_{n+i}$ is a chain with $n + i$ elements. This proves (2).

To prove (3) it suffices to show that (3) implies (2) since both $\{L_\alpha\}$ and $\{M_\alpha\}$ form bases of QSym. Assuming (3), we calculate

$$L_{\mathcal{C}(D)}^{(i)} = \sum_{C \supset D} M_{\mathcal{C}(C)}^{(i)} = \sum_{C \supset D} \sum_{B \subset [n+i-1]} |T(C,B)| \, M_{\mathcal{C}(B)}.$$

We show that

$$\sum_{C \supset D} \sum_{B \subset [n+i-1]} |T(C,B)| \, M_{\mathcal{C}(B)} = \sum_{E \subset [n+i-1]} |T(D,E)| \sum_{B \supset E} M_{\mathcal{C}(B)}$$

from which our claim will follow. A term $M_{\mathcal{C}(B)}$ on the right hand side is indexed by the following data: an i-extension $t : [n-1] \to [n+i-1]$ (of D) and the set $B \backslash E$ contained in $t([n-1] \backslash D)$. But since t is injective, this is the same as giving the subset $t^{-1}(B \backslash E) \subset [n-1] \backslash D$ and the i-extension $t : [n-1] \to [n+i-1]$. This is exactly the information indexing terms on the left hand side.

To prove the last claim, let $F \subset [n+i+j-1]$ and $t : [n-1] \to [n+i+j-1]$ be an $i + j$-extension of D to F. Now let $[n+i+j-1] \backslash t([n-1]) = S \cup S'$ be a decomposition into a set S containing j-elements and S' containing i-elements. Then there is a unique $E \subset [n+i-1]$ and $t' : [n+i-1] \to [n+i+j-1]$ such that t' is a j-extension of E to F satisfying $S = [n+i+j-1] \backslash t'([n+i-1])$. Furthermore, there is a unique $t'' : [n-1] \to [n+i-1]$ which is an i-extension of D to E such that $[n+i-1] \backslash t''([n-1]) = (t')^{-1}(S')$. The composition $t' \circ t''$ is equal to t. The

correspondence $(t, S) \mapsto (t', t'')$ is a bijection. Since there are exactly $\binom{i+j}{i}$ choices for S, this proves that $(f^{(i)})^{(j)} = \binom{i+j}{i} f^{(i+j)}$. $\square$

Example 5.13. Take $\alpha = (2, 1)$. Then $D(\alpha) = \{2\}$ and we have the following equality:
$$\tilde{L}_\alpha^{(2)} = L_{(1,1,2,1)} + 2L_{(1,2,1,1)} + 3L_{(2,1,1,1)}.$$
Here for example $|T(\{2\}, \{1, 3, 4\})| = 2$ since there are two maps $t_1 : \{1, 2\} \mapsto \{2, 3\}$ and $t_2 : \{1, 2\} \mapsto \{2, 4\}$ which satisfy the needed condition. By applying the rule we get the following part of decomposition of $\tilde{L}_{(2,1)}$ into L-s:
$$\tilde{L}_{(2,1)} = L_{(2,1)} + L_{(1,2,1)} + 2L_{(2,1,1)} + L_{(1,1,2,1)} + 2L_{(1,2,1,1)} + 3L_{(2,1,1,1)} + \ldots.$$

Remark 5.14. One may define the *multi-monomial quasisymmetric functions* $\tilde{M}_\alpha$ by analogy with usual monomial symmetric functions M_α:
$$\tilde{M}_\alpha = \sum_{D(\beta) \subset D(\alpha)} (-1)^{|D(\alpha)| - |D(\beta)|} \tilde{L}_\beta$$
where the summation is restricted to compositions satisfying $|\beta| = |\alpha|$. It is clear that $H_{|\alpha|}(\tilde{M}_\alpha) = M_\alpha$.

As an application of Theorem 5.12, we give a curious property of descents in the multi-Jordan-Holder set for partition shapes. It generalizes the following statement concerning usual Jordan-Holder sets.

Theorem 5.15. [17, Theorem 7.19.9] *Let λ/μ be a skew Young diagram with n boxes. For any $1 \leq i \leq n - 1$ the number $d_i(\lambda/\mu)$ of $w \in \mathcal{J}(\lambda/\mu, \theta_s)$ for which $i \in D(w)$ is independent of i.*

Theorem 5.15 is proved essentially using the following lemma, implicit in the argument of [17]. Define the linear transformation $\phi_i : \mathrm{QSym} \longrightarrow \mathbb{Z}$ by

$$\phi_i(L_\alpha) = \begin{cases} 1 & \text{if } i \in D(\alpha) \\ 0 & \text{otherwise,} \end{cases}$$

and extending by linearity.

Lemma 5.16. *Let $f = \sum c_\alpha M_\alpha \in \mathrm{QSym}$ be a homogeneous quasisymmetric function of degree n. Then $\phi_i(f)$ does not depend on i if and only if $c_{(2,1,\ldots,1)} = c_{(1,2,\ldots,1)} = \ldots = c_{(1,1,\ldots,2)}$.*

Proof. The statement follows from the equation $L_\alpha = \sum_{E \supset D(\alpha)} M_{\mathcal{C}(E)}$. $\square$

We call f satisfying the condition of Lemma 5.16 *balanced*. We also call the monomials labeled by the compositions $\alpha = (1^i, 2, 1^{n-2-i})$ *balancing*.

Lemma 5.17. *If f is a homogeneous balanced quasisymmetric function, then so is $f^{(i)}$ for any $i \geq 0$.*

Proof. The claim clearly holds for $i = 0$. Using the last statement of Theorem 5.12, we may assume that $i = 1$. Suppose f has degree n. Let $D_j = [n-1] \backslash \{j\}$ for $1 \leq j \leq n - 1$ and $E_k = [n] \backslash \{k\}$ for $1 \leq k \leq n$. By Theorem 5.12, to calculate the coefficient of $M_{\mathcal{C}(E_k)}$ in $f^{(1)}$ it suffices to find the 1-extensions $t : [n-i] \to [n]$ of some $D \subset [n-1]$ to E_k. But $|E_k| = n + 1$ so such D satisfy $|D| = n - 2$ and so must be of the form $D = D_j$ for some j. The number of 1-extensions of D_j to E_k

is equal to $n - k$ if $j = k$, equal to $k - 1$ if $j = k - 1$, and equal to 0 otherwise. By Lemma 5.16 the coefficient of $M_{\mathcal{C}(D_j)}$ in f does not depend on j, thus the coefficient of $M_{\mathcal{C}(E_k)}$ in $f^{(1)}$ is equal to $n - 1$ times the coefficient of $M_{\mathcal{C}(D_j)}$ in f, which does not depend on k. Again by Lemma 5.16, $f^{(1)}$ must be balanced. $\qquad\square$

We prove the following generalization of the Theorem 5.15.

Theorem 5.18. *Let $|\lambda/\mu| = n$. For any $N \geq n$ and $1 \leq i \leq N - 1$ the number $d_i(\lambda/\mu)$ of $w \in \tilde{\mathcal{J}}_N(\lambda/\mu, \theta_s)$ for which $i \in D(w)$ is independent of i.*

Proof. Let $G'_{\lambda/\mu} = \tilde{K}_{\lambda/\mu,\theta_s}$, which has lowest degree homogeneous component $H_0(G'_{\lambda/\mu}) = s_{\lambda/\mu}$ equal to a Schur function. We shall see later (Corollary 6.3)) that $G'_{\lambda/\mu}$ is a symmetric function (of unbounded degree). By Theorem 5.6 for each $N \geq n$ we have

$$H_N(G'_{\lambda/\mu}) = \sum_{n \leq m \leq N} \sum_{w \in \tilde{\mathcal{J}}_m(\lambda/\mu,\theta_s)} L^{(N-m)}_{\mathcal{C}(w)}.$$

We know that $s_{\lambda/\mu} = \sum_{w \in \tilde{\mathcal{J}}_n(\lambda/\mu,\theta_s)} L_{\mathcal{C}(w)}$ is symmetric, and thus balanced by Lemma 5.16. By Lemma 5.17, $s^{(1)}_{\lambda/\mu}$ is balanced and we also know $H_{n+1}(G'_{\lambda/\mu})$ is symmetric so $\sum_{w \in \tilde{\mathcal{J}}_{n+1}(\lambda/\mu,\theta_s)} L_{\mathcal{C}(w)} = H_{n+1}(G'_{\lambda/\mu}) - s^{(1)}_{\lambda/\mu}$ is also balanced. Proceeding in this manner we conclude that each of the sums $\sum_{w \in \tilde{\mathcal{J}}_N(\lambda/\mu,\theta_s)} L_{\mathcal{C}(w)}$ is balanced. By Lemma 5.16 we obtain exactly the needed result. $\qquad\square$

Example 5.19. One can check that in Example 5.8 for each $1 \leq i \leq 4$ there exists exactly two elements of $\tilde{\mathcal{J}}_5(\lambda, \theta_s)$ with i as a descent.

6. K-theory of Grassmannians and $\mathfrak{m}$Sym

6.1. **Fomin-Greene operators.** Let Λ denote the set of partitions as before. If $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l > 0)$ is a partition, then it contains the boxes (i, j) for $1 \leq i \leq l$ and $1 \leq j \leq \lambda_i$. The box (i, j) is on *diagonal* $j - i$. We say that λ has an *inner corner* on the i-th diagonal if there exists $\mu \in \Lambda$ such that λ/μ is a single box on the i-th diagonal. Similarly, λ has an *outer corner* on the i-th diagonal if there exists $\mu \in \Lambda$ such that μ/λ is a single box on the i-th diagonal.

Fix a partition $\nu \in \Lambda$. Let $\mathbb{Z}\Lambda_\nu = \oplus_{\nu \subset \lambda}\mathbb{Z} \cdot \lambda$ denote the free $\mathbb{Z}$-module with a basis of partitions containing ν, equipped with a non-degenerate pairing $\langle ., . \rangle$: $\mathbb{Z}\Lambda_\nu \times \mathbb{Z}\Lambda_\nu \to \mathbb{Z}$ defined by $\langle \lambda, \mu \rangle = \delta_{\lambda\mu}$. Now for each $i \in \mathbb{Z}$, define a $\mathbb{Z}$-linear operator $v_i : \mathbb{Z}\Lambda_\nu \to \mathbb{Z}\Lambda_\nu$ by

$$v_i \cdot \lambda = \begin{cases} \mu & \text{if } \lambda \text{ has an outer corner } \mu/\lambda \text{ on the } i\text{-th diagonal,} \\ \lambda & \text{if } \lambda \text{ has an inner corner not contained in } \nu \text{ on the } i\text{-th diagonal,} \\ 0 & \text{otherwise,} \end{cases}$$

and extending by linearity. The operators v_i satisfy the relations:

$$\begin{aligned} v_i^2 &= v_i & &\text{for each } i \in \mathbb{Z}, \\ v_i v_{i+1} v_i &= v_{i+1} v_i v_{i+1} = 0 & &\text{for each } i \in \mathbb{Z}, \\ v_i v_j &= v_j v_i & &\text{for each } i, j \in \mathbb{Z} \text{ with } |i - j| \geq 2. \end{aligned}$$

Operators very closely related to the v_i are studied by Fomin and Greene [4] and operators differing from ours by a sign are studied by Buch [3]. We briefly

explain their connection with set-valued P-partitions to draw an analogy with our construction of $\mathfrak{M}$Sym later. Define a formal power series

$$A(x) = \cdots (1 + xv_2)(1 + xv_1)(1 + xv_0)(1 + xv_{-1}) \cdots$$

with coefficients in operators on $\mathbb{Z}\Lambda_\nu$. The action of $A(x)$ on $a \in \mathbb{Z}\Lambda_\nu$ gives a well defined element of $\mathbb{Z}\Lambda_\nu[[x]]$. The following result is essentially [3, Theorem 3.1] with a sign omitted.

Lemma 6.1. *Let λ/μ be a skew partition. Then*

$$\tilde{K}_{\lambda/\nu, \theta_s}(x_1, x_2, \ldots) = \langle \cdots A(x_2)A(x_1) \cdot \nu, \lambda \rangle.$$

The next result follows from work of Fomin and Greene [4].

Lemma 6.2. *We have $A(x)A(y) = A(y)A(x)$ as operators on $\mathbb{Z}\Lambda_\nu$.*

Corollary 6.3. *Let λ/μ be a skew shape. Then $\tilde{K}_{\lambda/\mu, \theta_s}$ is a symmetric function (of unbounded degree).*

Proof. Follows immediately from Lemmas 6.1 and 6.2. $\qquad\square$

6.2. **The Hopf algebra $\mathfrak{m}$Sym and K-theory of Grassmannians.** Let $\mathfrak{m}$Sym $= \prod_\lambda \mathbb{Z} \cdot \tilde{K}_{\lambda, \theta_s}$ denote the subspace of $\mathfrak{m}$QSym continuously spanned by the generating functions $\tilde{K}_{\lambda, \theta_s}$ as λ varies over all partitions. For a fixed composition α, $\tilde{L}_\alpha$ only occurs in finitely many $\tilde{K}_{\lambda, \theta_s}$ so $\mathfrak{m}$Sym is indeed a subspace of $\mathfrak{m}$QSym. For convenience we now write $\tilde{K}_\lambda$ instead of $\tilde{K}_{\lambda, \theta_s}$. Also we shall call a set-valued $(\lambda/\mu, \theta_s)$-partition σ simply a *set-valued tableau* of shape λ/μ.

Proposition 6.4. *The space $\mathfrak{m}$Sym is a Hopf subalgebra of $\mathfrak{m}$QSym. It is isomorphic to the completion $\widehat{\mathrm{Sym}} = \prod_\lambda \mathbb{Z} \cdot s_\lambda$ of the algebra of symmetric functions.*

Proof. Since the lowest degree homogeneous component of $\tilde{K}_\lambda$ is equal to the Schur function s_λ, the space $\mathfrak{m}$Sym is equal to the space $\prod_\lambda \mathbb{Z} \cdot s_\lambda$ of arbitrary linearly combinations of Schur functions. Thus $\mathfrak{m}$Sym is the Hopf subalgebra of $\mathfrak{m}$QSym consisting of symmetric functions of unbounded degree. $\qquad\square$

In [3], Buch studied a bialgebra $\Gamma = \oplus_\lambda \mathbb{Z} \cdot G_\lambda$ spanned by the set $\{G_\lambda\}$ of stable Grothendieck polynomials. The stable Grothendieck polynomials were first studied by Fomin and Kirillov [5] and defined as stable limits of Grothendieck polynomials [12]. For our purposes, they can be defined as follows.

Theorem 6.5. [3, Theorem 3.1] *The stable Grothendieck polynomial $G_{\nu/\lambda}(x)$ is given by the formula*

$$G_{\nu/\lambda}(x) = \sum_T (-1)^{|T| - |\nu/\lambda|} x^T,$$

where the sum is taken over all set-valued tableaux T of shape ν/λ.

Thus the symmetric function G_λ can be obtained from $\tilde{K}_\lambda$ by changing the degree n homogeneous component of $\tilde{K}_\lambda$ by the sign $(-1)^{n-|\lambda|}$, or in other words one has $\tilde{K}_\lambda(x_1, x_2, \ldots) = (-1)^{|\lambda|} G_\lambda(-x_1, -x_2, \ldots)$.

Buch related the structure constants of Γ to the K-theory $K^\circ \mathrm{Gr}(k, \mathbb{C}^n)$ of the Grassmannian $\mathrm{Gr}(k, \mathbb{C}^n)$ of k-planes in $\mathbb{C}^n$. In addition, Buch described the structure constants completely using the combinatorics of set-valued tableaux. We briefly describe the connections with K-theory here and return to the combinatorial descriptions from a dual point of view later (see Section 9.5).

Let $R = (n-k)^k$ denote the the rectangle with k rows and $n-k$ columns and let I_R be the subspace of Γ spanned by G_λ for λ not contained in R. Let $K^\circ \mathrm{Gr}(k, \mathbb{C}^n)$ denote the Grothendieck group of algebraic vector bundles on $\mathrm{Gr}(k, \mathbb{C}^n)$. It is naturally isomorphic to the Grothendieck group $K_\circ \mathrm{Gr}(k, \mathbb{C}^n)$ of coherent sheaves on $\mathrm{Gr}(k, \mathbb{C}^n)$. The K-group $K^\circ \mathrm{Gr}(k, \mathbb{C}^n)$ is spanned by the classes $[\mathcal{O}_\lambda]$ of structure sheaves of Schubert varieties X_λ indexed by partitions $\lambda \subset R$. For convenience we set $[\mathcal{O}_\lambda] = 0$ if λ does not fit inside R. The K-theory $K^\circ \mathrm{Gr}(k, \mathbb{C}^n)$ becomes a commutative ring when equipped with the multiplication induced by tensor products of vector bundles.

Theorem 6.6 ([3, Theorem 8.1]). *The map $G_\lambda \mapsto [\mathcal{O}_\lambda]$ induces an isomorphism of rings $\Gamma/I_R \simeq K^\circ \mathrm{Gr}(k, \mathbb{C}^n)$.*

Note that Γ/I_R is isomorphic to the quotient of $\mathfrak{m}\mathrm{Sym}$ by the continuous span $\prod_{\lambda \not\subseteq R} \tilde{K}_\lambda$ since both spaces are finite-dimensional. To explain the geometric meaning of the coproduct, fix $k_1 < n_1$ and $k_2 < n_2$. Taking the direct sum of vector spaces induces a map $\phi : \mathrm{Gr}(k_1, \mathbb{C}^{n_1}) \times \mathrm{Gr}(k_2, \mathbb{C}^{n_2}) \to \mathrm{Gr}(k_1 + k_2, \mathbb{C}^{n_1 + n_2})$. Then Buch shows that $\phi^*([\mathcal{O}_\lambda]) = \sum_{\mu,\nu} d_\lambda^{\mu\nu} [\mathcal{O}_\mu] \otimes [\mathcal{O}_\nu]$ where we have identified $K^\circ \mathrm{Gr}(k_1, \mathbb{C}^{n_1}) \times \mathrm{Gr}(k_2, \mathbb{C}^{n_2})$ with $K^\circ \mathrm{Gr}(k_1, \mathbb{C}^{n_1}) \otimes K^\circ \mathrm{Gr}(k_2, \mathbb{C}^{n_2})$. Here the $d_\lambda^{\mu\nu}$ are the structure constants of the coproduct: $\Delta G_\lambda = \sum_{\mu,\nu} d_\lambda^{\mu\nu} G_\mu \otimes G_\nu$.

7. The big Multi Malvenuto-Reutenauer Hopf algebra

7.1. **Big Multi-permutations and set compositions.**

Definition 7.1. A *big Multi-permutation* or $\mathfrak{M}$-permutation of $[n]$ is a word $w = w_1 w_2 \cdots w_k$ such that (a) each w_i is a subset of $\mathbb{P}$ not containing consecutive numbers and (b) the disjoint union $\sqcup_{i=1}^k w_i$ is equal to the set $[n]$. We say that w has length $\ell(w) = k$.

Denote the set of $\mathfrak{M}$-permutations of $[n]$ by $S_n^{\mathfrak{M}}$ and let $S_\infty^{\mathfrak{M}} = \cup_n S_n^{\mathfrak{M}}$. For example, we have $w = [(1,3), (5,7,9), (10), (4,6), 2, 8] \in S_{10}^{\mathfrak{M}}$ which has length $\ell(w) = 6$. By convention $S_0^{\mathfrak{M}}$ contains a single element – the empty $\mathfrak{M}$-permutation $\emptyset$. We let $\mathfrak{M}\mathrm{MR} = \oplus_{w \in S^{\mathfrak{M}}} \mathbb{Z}.w$ denote the free $\mathbb{Z}$-module of finite $\mathbb{Z}$-linear combinations of $\mathfrak{M}$-permutations.

Recall that a *set composition* $w = w_1 w_2 \cdots w_k$ of a finite set S is a word $w = w_1 w_2 \cdots w_k$ such that the disjoint union $\sqcup_{i=1}^k w_i$ is equal to the set S. Thus w_i may contain consecutive numbers. If $w = w_1 w_2 \cdots w_k$ is a set composition of $S \subset \mathbb{P}$, we define the *standardization* $\mathrm{st}(w) \in S_\infty^{\mathfrak{M}}$ by repeatedly doing the following operations until one has a $\mathfrak{M}$-permutation:

   R1 delete the letter $i+1$ if both i and $i+1$ belong to some w_j, and
   R2 reduce all letters larger than i by 1 if i is not present in any w_j.

It is clear that $\mathrm{st}(w)$ is a well defined $\mathfrak{M}$-permutation. Also if w is a set composition of S and $T \subset S$ then the *restriction* $w|_T$ of w to T is obtained by intersecting each w_i with T and removing all the w_j which become empty. The restriction $w|_T$ is a set composition of T.

Let $w = w_1 \cdots w_k \in S_m^{\mathfrak{M}}$ and $u = u_1 \cdots u_l \in S_n^{\mathfrak{M}}$. Define a product $\bullet$ on $\mathfrak{M}\mathrm{MR}$ by extending the formula

$$w \bullet u = \sum v$$

by linearity, where the (finite) sum is taken over

(1) all $v \in S_{m+n}^{\mathfrak{M}}$ such that $v|_{[m]} = w$ and $\mathrm{st}(v|_{[m+1,m+n]}) = u$; and
(2) all $v \in S_{m+n-1}^{\mathfrak{M}}$ such that $v|_{[m]} = w$ and $\mathrm{st}(v|_{[m,m+n-1]}) = u$.

Proposition 7.2. *The product* $\bullet : \mathfrak{M}\mathrm{MR} \otimes \mathfrak{M}\mathrm{MR} \to \mathfrak{M}\mathrm{MR}$ *is associative.*

Proof. Let $w = w_1 \cdots w_k \in S_m^{\mathfrak{M}}$, $u = u_1 \cdots u_l \in S_n^{\mathfrak{M}}$, and $x = x_1 \cdots x_r \in S_p^{\mathfrak{M}}$. Then one checks that both $(w \bullet u) \bullet x$ and $w \bullet (u \bullet x)$ are equal to the sum over

(1) all $v \in S_{m+n+p}^{\mathfrak{M}}$ such that $v|_{[m]} = w$, $\mathrm{st}(v|_{[m+1,m+n]}) = u$ and
$\mathrm{st}(v|_{[m+n+1,m+n+p]}) = x$; and
(2) all $v \in S_{m+n+p-1}^{\mathfrak{M}}$ such that $v|_{[m]} = w$, $\mathrm{st}(v|_{[m,m+n-1]}) = u$ and
$\mathrm{st}(v|_{[m+n,m+n+p-1]}) = x$; and
(3) all $v \in S_{m+n+p-1}^{\mathfrak{M}}$ such that $v|_{[m]} = w$, $\mathrm{st}(v|_{[m+1,m+n]}) = u$ and
$\mathrm{st}(v|_{[m+n,m+n+p-1]}) = x$; and
(4) all $v \in S_{m+n+p-2}^{\mathfrak{M}}$ such that $v|_{[m]} = w$, $\mathrm{st}(v|_{[m,m+n-1]}) = u$ and
$\mathrm{st}(v|_{[m+n-1,m+n+p-2]}) = x$.

If $v \in S_{m+n+p-1}^{\mathfrak{M}}$ satisfies both the conditions in (2) and (3) then it occurs with multiplicity two in the product of w, u and x. $\qquad\square$

We can also give an alternative recursive definition of the product $\bullet$. For two set compositions a, b let $[a, b]$ denote their concatenation (assuming that the result is a set composition). We can extend $[., .]$ by linearity by distributing it over addition. We first define the semishuffle product $\circ$ as follows. Let $u = u_1 u'$ and $v = v_1 v'$ be $\mathfrak{M}$-permutations where u_1, v_1 are sets and u', v' are set compositions. Then

$$u \circ v = [u_1, u' \circ v] + [v_1, u \circ v'] + [u_1 \cup v_1, u' \circ v'].$$

For a $\mathfrak{M}$-permutation v, let $(v+n)$ denote the set composition obtained by increasing every number in v by n. Let $u \in S_n^{\mathfrak{M}}$.

Proposition 7.3. *We have* $u \bullet v = \mathrm{st}(u \circ (v + n))$.

Now define the coproduct $\triangle$ on $\mathfrak{M}\mathrm{MR}$ by

$$\triangle w = \sum_{[u,v]=w} \mathrm{st}(u) \otimes \mathrm{st}(v),$$

where the sum is over all pairs (u, v) of (possibly empty) set compositions which concatenate to w. We extend $\triangle$ by linearity to give $\triangle \colon \mathfrak{M}\mathrm{MR} \to \mathfrak{M}\mathrm{MR} \otimes \mathfrak{M}\mathrm{MR}$. The following result is immediate from the definition.

Proposition 7.4. *The coproduct* $\triangle \colon \mathfrak{M}\mathrm{MR} \otimes \mathfrak{M}\mathrm{MR} \to \mathfrak{M}\mathrm{MR}$ *is coassociative.*

Define the unit map $\eta : \mathbb{Z} \to \mathfrak{M}\mathrm{MR}$ by $\eta(1) = \emptyset$ and the counit map $\varepsilon : \mathfrak{M}\mathrm{MR} \to \mathbb{Z}$ by taking the coefficient of $\emptyset$.

Theorem 7.5. *The space* $\mathfrak{M}\mathrm{MR}$ *is a bialgebra with product* $\bullet$, *coproduct* $\triangle$, *unit* η, *and counit* ε.

Proof. It is easy to check that $\mathfrak{M}\mathrm{MR}$ is a unital associative algebra and a counital coassociative coalgebra. We must therefore check that $\bullet$ and $\triangle$ are compatible. Let $w = w_1 \cdots w_k \in S_m^{\mathfrak{M}}$ and $u = u_1 \cdots u_l \in S_n^{\mathfrak{M}}$. Then both $\triangle (w \bullet u)$ and $(\triangle w) \bullet (\triangle u)$ are sums over terms $a \otimes b$ which can be described as follows. First let u' be the unique set composition of $[m + 1, m + n]$ such that $\mathrm{st}(u') = u$. Then $a = \mathrm{st}(a' = a_1' a_2' \cdots a_r')$ where for each $1 \le i \le r$, we have a_i' is either (a) equal to some w_j, or (b) equal to some u_s', or (c) the union of w_j and u_s'. Also if w_{j_1} appears

in a'_{i_1} and w_{j_2} in a'_{i_2} then $i_1 < i_2 \Rightarrow j_1 < j_2$ and furthermore all w_j with $j < j_2$ have to appear in a' (and the analogous statement for u'). Similarly $b = \mathrm{st}(b')$ for an analogously described b', and in addition the disjoint union of all the numbers in a' and b' is equal to $[m + n]$. $\square$

Call an element $w \in S_\infty^{\mathfrak{M}}$ *irreducible* if it cannot be written in the form $w = v/u = v_1 \ldots v_k (u_1 + n) \ldots (u_l + n)$ for two smaller (that is, non-empty) $\mathfrak{M}$-permutations $v, u \in S_\infty^{\mathfrak{M}}$. Let $\mathrm{Irr}^{\mathfrak{M}}$ be the set of irreducible $\mathfrak{M}$-permutations.

Theorem 7.6. *The algebra $S_\infty^{\mathfrak{M}}$ is free over the set of its irreducible elements.*

Proof. Any $\mathfrak{M}$-permutation $w \in S_\infty^{\mathfrak{M}}$ can be uniquely written as $w^1/w^2/\cdots/w^k$ where the $w^i \in \mathrm{Irr}^{\mathfrak{M}}$ are irreducible; we say that w is k-reducible. Thus the tensor space $\bigoplus_{n \geq 0} \mathbb{Z}(\mathrm{Irr}^{\mathfrak{M}})^{\otimes n}$ is (naturally) isomorphic as a free $\mathbb{Z}$-module to $\mathfrak{M}\mathrm{MR}$.

Now if $w^1, \ldots, w^k \in \mathrm{Irr}^{\mathfrak{M}}$ are irreducible then it follows from the definition of $\bullet$ that the only k-reducible term in $w^1 \bullet w^2 \bullet \cdots \bullet w^k$ is $w^1/w^2/\cdots/w^k$. This implies (via a triangularity argument) that the map $w^1 \otimes w^2 \otimes \cdots \otimes w^k \mapsto w^1 \bullet w^2 \bullet \cdots \bullet w^k$ induces a surjective algebra homomorphism $\bigoplus_{n \geq 0} \mathbb{Z}(\mathrm{Irr}^{\mathfrak{M}})^{\otimes n} \to \mathfrak{M}\mathrm{MR}$. The previous paragraph implies that this surjective map is an isomorphism.

$\square$

Remark 7.7. The Hopf structure of $\mathfrak{M}\mathrm{MR}$ may be related to the Hopf algebra of set partitions defined in [14].

7.2. **The antipode of $\mathfrak{M}\mathrm{MR}$.** We show that $\mathfrak{M}\mathrm{MR}$ has an antipode via a general construction following [2, Section 5] (see also [18]). Let H be any bialgebra with multiplication m, coproduct Δ, unit μ and counit ε. For each $i \geq 1$ let $m^{(i)} : H^{\otimes i+1} \to H$ denote the i-fold iterated product (by associativity the order does not matter) and let $\Delta^{(i)} : H \to H^{\otimes i+1}$ denote the i-fold iterated coproduct. In addition we set $m^{(0)} = \Delta^{(0)} = \mathrm{id} : H \to H$, $m^{(-1)} = \mu$, and $\Delta^{(-1)} = \varepsilon$. If $f : H \to H$ is any linear map then its i-fold convolution is $f^{(i)} = m^{(i-1)} f^{\otimes i} \Delta^{(i-1)}$.

Now set $\pi = \mathrm{id} - \mu\varepsilon$. If π is locally nilpotent with respect to convolution then the antipode $S : H \to H$ is given by

$$(4) \qquad S = \sum_{i \geq 0} (-1)^i m^{(i-1)} \pi^{\otimes i} \Delta^{(i-1)}.$$

Proposition 7.8. *The bialgebra $\mathfrak{M}\mathrm{MR}$ has an antipode, making it a Hopf-algebra.*

Proof. By the preceding discussion it suffices to show that π is locally nilpotent. Let $a \in \mathfrak{M}\mathrm{MR}$ be non-zero. Let $n \geq 0$ be the maximal value such that some w satisfying $\ell(w) = n$ occurs in a with non-zero coefficient. Then using the definition of $\triangle$, each term in $\triangle^{(n)} a$ must involve $\emptyset$ in one of its factors. But $\pi(\emptyset) = 0$ so $\pi^{(n+1)} a = 0$. $\square$

7.3. **Weak order on $\mathfrak{M}$-permutations.** The results of this section suggest that there may be a polytopal structure on $\mathfrak{M}$-permutations. Let $\mathrm{SC}(n)$ denote the collection of set compositions of $[n]$ and let $\mathrm{SC}(\infty) = \cup_{n \geq 1} \mathrm{SC}(n)$. In [9] set compositions were considered under the name of *pseudo permutations*, while in [14] they are identified with faces of permutohedra. In [9] and [14] independently the following analog of the weak order is defined on $\mathrm{SC}(n)$, for each $n \geq 1$. Let $w = w_1 w_2 \cdots w_k \in \mathrm{SC}(n)$. The covers of w can be completely described in the following manner.

(1) if $w_i < w_{i+1}$ then $w \lessdot w_1 \cdots w_{i-1}(w_i \cup w_{i+1})w_{i+1} \cdots w_k$;

(2) if $w_i = w_i' \sqcup w_i''$ is a disjoint union of non-empty subsets and $w_i' < w_i''$ then $w \lessdot w_1 \cdots w_{i-1}w_i''w_i'w_{i+1} \cdots w_k$.

The order $\prec$ on $\mathrm{SC}(n)$ is then the transitive closure of $\lessdot$.

Now let $\sim$ be the equivalence relation on $\mathrm{SC}(\infty)$ obtained by taking the transitive closure of the relations $w \sim' \mathrm{st}(w)$ for each $w \in \mathrm{SC}(\infty)$. Thus each equivalence class contains a unique $\mathfrak{M}$-permutation. For example,

$$[(1,4,5),7,(2,8,9),(6,10),3] \sim [(1,4),(6,7,8),(2,9),(5,10,11),3]$$

since

$$\mathrm{st}([(1,4,5),7,(2,8,9),(6,10),3]) = \mathrm{st}([(1,4),(6,7,8),(2,9),(5,10,11),3])$$
$$= [(1,4),6,(2,7),(5,8),3].$$

If $w \sim u$ are two equivalent set compositions, let us say that w *contains* u if u can be obtained from w by partially standardizing: in other words, by using the operations (R1) and (R2) of Section 7.1 a number of times. The following two lemmata are easy to establish from the definitions.

Lemma 7.9. *If $w \sim u$ are two set compositions then there exists a set composition v equivalent to both which contains both.*

Lemma 7.10. *Suppose $w \lessdot v$ is a cover relation in $\mathrm{SC}(n)$. If w' contains w then there is a canonical cover relation $w' \lessdot v'$ such that v' contains v.*

Now define the *weak order* $<$ on $S_\infty^{\mathfrak{M}}$ by taking the transitive closure of the following relations: $w \in S_\infty^{\mathfrak{M}}$ is less than $v \in S_\infty^{\mathfrak{M}}$ if there exists set compositions w' and v' so that $w' \sim w$, $v' \sim v$ and $w' \lessdot v'$.

Theorem 7.11. *The weak order on $S_\infty^{\mathfrak{M}}$ is a valid partial order.*

Proof. We need to show that for two $\mathfrak{M}$-permutations w, w' that if $w < w'$ and $w' < w$ then $w = w'$. Alternatively we need to show that there is no sequence $w_1, w_2, \ldots, w_n, w_1$ of $\mathfrak{M}$-permutations such that $v_1 \lessdot u_2, v_2 \lessdot u_3, \cdots v_{n-1} \lessdot u_n, v_n \lessdot u_1$ for some set compositions $v_i \sim w_i$ and $u_i \sim w_i$. Using Lemma 7.9 and Lemma 7.10 repeatedly, we may assume that $v_i = u_i$ for $i \neq 1$. Thus we are reduced to proving that if $v < u$ in $\mathrm{SC}(n)$ for some n then $\mathrm{st}(v) \neq \mathrm{st}(u)$.

Suppose that $v < u$ and $\mathrm{st}(v) = \mathrm{st}(u) = w$ for some $\mathfrak{M}$-permutation $w \in S_m^{\mathfrak{M}}$. Define the "standardization" function $f_v : [n] \to [m]$ by requiring that (a) f_v is increasing, and (b) $f_v^{-1}(i)$ is a non-empty interval completely contained in the set v_k if $i \in w_k$. Similarly define f_u. Now since $\mathrm{st}(v) = \mathrm{st}(u) = w$ we must have that $v = v_1 \cdots v_k$ and $u = u_1 \cdots u_k$ have the same length. Thus it makes sense to ask for the smallest integer i such that $i \in v_j$ and $i \in u_k$ for $j \neq k$. But since $\mathrm{st}(v) = \mathrm{st}(u)$, the letter $i - 1$ must be either in both v_j and u_j or in both v_k and u_k. Let us say that a set composition x has an *inversion* at (i,j) for $i < j$ if j precedes i; and a *half-inversion* if j and i belong to the same set of x. We note that weak order on $\mathrm{SC}(n)$ either increases or does not change inversions for each (i,j).

Now suppose $j < k$. If $i - 1 \in v_j \cap u_j$ then v has a half-inversion at $(i-1,i)$ while u has no inversion which is impossible. If $i - 1 \in v_k \cap u_k$ then v has an inversion at $(i-1,i)$ while u only has a half-inversion which again is impossible. Thus $j > k$.

Suppose first that $i - 1 \in v_k \cap u_k$. Let i_1 be the smallest integer greater than i such that $i_1 \notin u_k$. Then $i_1 \in f_u^{-1}(f_v(i))$ so $i_1 \in u_j$. Since u has no inversion

at (i, i_1), it must also be the case that v has no inversion at (i, i_1). Thus $i_1 \in v_{j_1}$ for $j_1 > j$. Now let i_2 be the smallest integer in $f_u^{-1}(f_v(i_1))$. Clearly $i_2 > i_1$ and $i_1 \in u_j$ while $i_2 \in u_{j_1}$. Again u has no inversion at (i_1, i_2) so v has no inversion at (i_1, i_2) and we deduce that $i_2 \in v_{j_2}$ for $j_2 > j_1$. Continuing in this manner we produce an infinite sequence $i_1, i_2, \ldots$. Since n is finite, we arrive at a contradiction.

The case $i - 1 \in v_j \cap u_j$ is similar. $\square$

7.4. **Duality between $\mathfrak{m}$MR and $\mathfrak{M}$MR.** Let V be the space of infinite $\mathbb{Z}$-linear combinations of elements $\{v_s \mid s \in S\}$ where S is some indexing set. As in Section 4 call a linear functional $f : V \to \mathbb{Z}$ continuous if it respects infinite linear combinations, not just finite ones. The set of all continuous linear functionals forms the *continuous dual* V_c^* of V and is a free $\mathbb{Z}$-module with basis $\{f_s \mid s \in S\}$ where f_s is defined by $f_s(v_{s'}) = \delta_{ss'}$. The $\mathbb{Z}$-module V is then the usual dual of V_c^*. Abusing notation slightly, we may simply say that V and V_c^* are *continuous duals* with dual bases $\{v_s \mid s \in S\}$ and $\{f_s \mid s \in S\}$ (even though $\{v_s\}$ may *not* be a basis of V). This notion of continuous duality makes sense for bialgebras, and Hopf algebras, with distinguished bases.

There is a natural way to consider $\mathfrak{M}$-permutations and $\mathfrak{m}$-permutations to be inverses of each other. If $w = w_1 \cdots w_k \in S_n^{\mathfrak{M}}$ then $u = w^{-1}$ is the $\mathfrak{m}$-permutation of k with length n such that the i-s in u occur in the positions specified by w_i. Clearly this gives rise to a bijection between $\mathfrak{M}$-permutations and $\mathfrak{m}$-permutations. For example, if w is the $\mathfrak{m}$-permutation

$$[1, 5, 1, 4, 2, 4, 2, 6, 2, 3]$$

then w^{-1} is the $\mathfrak{M}$-permutation

$$[(1, 3), (5, 7, 9), (10), (4, 6), 2, 8].$$

We note that the two standardization operators are compatible in the following manner: $\mathrm{st}(w) = v$ if and only if $\mathrm{st}(w^{-1}) = v^{-1}$, where we have extended the inverse operation to words and set-compositions. Furthermore, if $v \in S_n^{\mathfrak{m}}$ then $v^{-1} \in S_{\ell(v)}^{\mathfrak{M}}$ and $\ell(v^{-1}) = n$.

Theorem 7.12. *The bialgebras $\mathfrak{M}$MR and $\mathfrak{m}$MR are continuous duals of each other with dual bases $\{w \mid w \in S_\infty^{\mathfrak{m}}\}$ and $\{v \mid v \in S_\infty^{\mathfrak{M}}\}$, where $S_\infty^{\mathfrak{m}}$ is identified with $S_\infty^{\mathfrak{M}}$ via $w \leftrightarrow w^{-1}$. The antipode $S_{\mathfrak{M}\mathrm{MR}}$ of $\mathfrak{M}$MR induces an antipode $S_{\mathfrak{m}\mathrm{MR}}$ of $\mathfrak{m}$MR making $\mathfrak{M}$MR and $\mathfrak{m}$MR continuous duals as Hopf algebras.*

Proof. The preceding comments on continuous duals allow us to prove the theorem by simply comparing structure constants. We first show that product structure constants of $\mathfrak{m}$MR are equal to coproduct structure constants of $\mathfrak{M}$MR. Let $u \in S_n^{\mathfrak{m}}$, $v \in S_m^{\mathfrak{m}}$ and $w \in S_{m+n}^{\mathfrak{m}}$. Since $*$ and $\triangle$ are multiplicity free, we must show that w occurs in $u * v$ if and only if $u^{-1} \otimes v^{-1}$ occurs in $\triangle w^{-1}$. Suppose that w occurs in $u * v$. If $w^{-1} = w_1' w_2' \cdots w_{n+m}'$ then $u^{-1} = \mathrm{st}(w_1' \cdots w_n')$ and $v^{-1} = \mathrm{st}(w_{n+1}' \cdots w_{n+m}')$. The converse is also clear. Similarly, one checks that w^{-1} occurs in $u^{-1} \bullet v^{-1}$ if and only if $u \otimes v$ occurs in Δw.

Finally, we need to check that that the antipode $S_{\mathfrak{M}\mathrm{MR}} : \mathfrak{M}\mathrm{MR} \to \mathfrak{M}\mathrm{MR}$ induces a well defined map $S_{\mathfrak{m}\mathrm{MR}} = S_{\mathfrak{M}\mathrm{MR}}^* : \mathfrak{m}\mathrm{MR} \to \mathfrak{m}\mathrm{MR}$ (here $S_{\mathfrak{M}\mathrm{MR}}^*$ denotes the continuous transpose). Let $w \in S_n^{\mathfrak{M}}$. Then by (4), $S_{\mathfrak{M}\mathrm{MR}}(w)$ is a linear combination of the basis elements $u \in \cup_{i \leq n} S_i^{\mathfrak{M}}$. Thus the continuous transpose $S_{\mathfrak{M}\mathrm{MR}}^* : \mathfrak{m}\mathrm{MR} \to \mathfrak{m}\mathrm{MR}$ sends the basis element $x = u^{-1} \in S_k^{\mathfrak{m}}$ to a (possibly infinite) linear combination of

the basis elements $\{y \mid \ell(y) \geq \ell(x)\}$. We need to show that this extends via continuous linearity to a well-defined map for arbitrary elements of $\mathfrak{m}$MR. But given a fixed $y \in S_\infty^{\mathfrak{m}}$ there are only finitely many $x \in S_\infty^{\mathfrak{m}}$ with shorter length. Thus the coefficient of y in $S^*_{\mathfrak{m}\mathrm{MR}}(\sum_{x \in S_\infty^{\mathfrak{m}}} a_x\, x)$ is well-defined. $\square$

8. The Hopf algebra of Multi-noncommutative symmetric functions

Now we define a Hopf subalgebra of $\mathfrak{M}$MR which will turn out to be the dual Hopf algebra of $\mathfrak{m}$QSym. Let α be a composition of n. We say that $w \in S_n^{\mathfrak{M}}$ is of *type* $\mathrm{type}(w) = \alpha$ if $w^{-1} \in S_\infty^{\mathfrak{m}}$ has descent set $\mathrm{Des}(w^{-1}) = D(\alpha)$. Alternatively, the type of w is $\mathcal{C}(D)$ where

$$D = \{i \in [1, n-1] \mid i+1 \text{ lies to the right of } i \text{ in } w\}.$$

Let $\tilde{R}_\alpha \in \mathfrak{M}$MR denote the sum of all $w \in S_n^{\mathfrak{M}}$ of type α. For example, if $\alpha = (3, 1)$ then

$$\tilde{R}_\alpha = [(1,4), 2, 3] + [1, (2,4), 3] + [1, 2, 4, 3] + [1, 4, 2, 3] + [4, 1, 2, 3].$$

Let $\mathfrak{M}$NSym denote the subspace of $\mathfrak{M}$MR spanned by the elements $\tilde{R}_\alpha$ as α varies over all compositions. Clearly the elements $\tilde{R}_\alpha$ are independent.

8.1. **The product structure on $\mathfrak{M}$NSym.** For two compositions $\alpha = (\alpha_1, \ldots, \alpha_k)$ and $\beta = (\beta_1, \ldots, \beta_l)$, recall the definitions of $\alpha \rhd \beta$ and $\alpha \lhd \beta$ from Section 2.3. We now define $\alpha \cdot \beta = (\alpha_1, \ldots, \alpha_k + \beta_1 - 1, \ldots, \beta_l)$. For example, if $\alpha = (3, 2, 5, 1)$ and $\beta = (4, 2)$ then $\alpha \rhd \beta = (3, 2, 5, 5, 2)$, $\alpha \lhd \beta = (3, 2, 5, 1, 4, 2)$ and $\alpha \cdot \beta = (3, 2, 5, 4, 2)$.

Proposition 8.1. *Let α be a composition of m and β be a composition of n. We have*

$$\tilde{R}_\alpha \bullet \tilde{R}_\beta = \tilde{R}_{\alpha \rhd \beta} + \tilde{R}_{\alpha \lhd \beta} + \tilde{R}_{\alpha \cdot \beta}.$$

Thus $\mathfrak{M}$NSym is closed under the product $\bullet$ of $\mathfrak{M}$MR.

Proof. Directly from the definition it is clear that $\tilde{R}_\alpha \bullet \tilde{R}_\beta$ is a multiplicity free sum of certain $\mathfrak{M}$-permutations $u \in S_{m+n}^{\mathfrak{M}} \cup S_{m+n-1}^{\mathfrak{M}}$. The type of $u \in S_{m+n}^{\mathfrak{M}}$ is determined by $\mathrm{type}(u|_{[m]})$, $\mathrm{type}(\mathrm{st}(u|_{[m+1,m+n]}))$ and whether m lies in front of $m+1$. Such a u occurs in the product $\tilde{R}_\alpha \bullet \tilde{R}_\beta$ if and only if $\mathrm{type}(u|_{[m]}) = \alpha$ and $\mathrm{type}(\mathrm{st}(u|_{[m+1,m+n]})) = \beta$. The type of $u \in S_{m+n-1}^{\mathfrak{M}}$ is determined by $\mathrm{type}(u|_{[m]})$ and $\mathrm{type}(\mathrm{st}(u|_{[m,m+n-1]}))$ and occurs in $\tilde{R}_\alpha \bullet \tilde{R}_\beta$ if and only if $\mathrm{type}(u|_{[m]}) = \alpha$ and $\mathrm{type}(\mathrm{st}(u|_{[m,m+n-1]})) = \beta$. The three terms $\tilde{R}_{\alpha \rhd \beta}$, $\tilde{R}_{\alpha \lhd \beta}$, and $\tilde{R}_{\alpha \cdot \beta}$ correspond (in order) to the following three possibilities for u: (a) $u \in S_{m+n}^{\mathfrak{M}}$ and m occurs before $m+1$, (b) $u \in S_{m+n}^{\mathfrak{M}}$ and m occurs after $m+1$, and (c) $u \in S_{m+n-1}^{\mathfrak{M}}$. $\square$

Proposition 8.2. *The algebra $\mathfrak{M}$NSym is isomorphic to the free algebra on the symbols $\tilde{R}_\alpha$ for each composition α, with relations $\tilde{R}_\alpha \bullet \tilde{R}_\beta = \tilde{R}_{\alpha \rhd \beta} + \tilde{R}_{\alpha \lhd \beta} + \tilde{R}_{\alpha \cdot \beta}$.*

Proof. We must show that the relation $\tilde{R}_\alpha \bullet \tilde{R}_\beta = \tilde{R}_{\alpha \rhd \beta} + \tilde{R}_{\alpha \lhd \beta} + \tilde{R}_{\alpha \cdot \beta}$ implies all possible relations amongst the elements $\{R_\alpha\}$. Assume we have a relation in $\mathfrak{M}$NSym. Using $\tilde{R}_\alpha \bullet \tilde{R}_\beta = \tilde{R}_{\alpha \rhd \beta} + \tilde{R}_{\alpha \lhd \beta} + \tilde{R}_{\alpha \cdot \beta}$ we may make the relation linear. If all terms cancel out we conclude that the original relation is implied by our basic set of relations. Otherwise we obtain a linear dependence amongst the $\tilde{R}_\alpha$'s. But this is impossible as the sets of $\mathfrak{M}$-permutations involved in each $\tilde{R}_\alpha$ are disjoint, and the set of $\mathfrak{M}$-permutations forms a basis of $\mathfrak{M}$MR. $\square$

Define $F_k = \tilde{R}_{(k)} = [1, 2, \ldots, k] \in \mathfrak{M}\mathrm{NSym}$.

Proposition 8.3. $\mathfrak{M}\mathrm{NSym}$ *is freely generated over* $\mathbb{Z}$ *by* $\{F_k \mid k \geq 1\}$ *as an algebra.*

Proof. First we show that any $\tilde{R}_\alpha$ can be written as polynomial in the F_k's. The proof proceeds by induction on the number of parts in α. The base case $\alpha = (k)$ is clear.

Let $\alpha = (\alpha_1, \ldots, \alpha_l)$ with $l \geq 2$ and let $\bar{\alpha} = (\alpha_1, \ldots, \alpha_{l-1})$. Using Proposition 8.1 to write $\tilde{R}_\alpha = \tilde{R}_{\bar{\alpha}} F_{\alpha_l} - \tilde{R}_{\alpha'} - \tilde{R}_{\alpha''}$ where $\alpha' = \bar{\alpha} \rhd \alpha_l = (\alpha_1, \ldots, \alpha_{l-2}, \alpha_{l-1} + \alpha_l)$ and $\alpha'' = \bar{\alpha} \cdot \alpha_l = (\alpha_1, \ldots, \alpha_{l-2}, \alpha_{l-1} + \alpha_l - 1)$. Since $\tilde{R}_{\bar{\alpha}}$, $\tilde{R}_{\alpha'}$ and $\tilde{R}_{\alpha''}$ all have less parts than α by the inductive assumption we may assume that they can be expressed in terms of the F_k's; therefore $\tilde{R}_\alpha$ can also be written in terms of the F_k's.

Now we want to show that F_k's are algebraically independent. Assume that is not so and we have a non-trivial polynomial relation r. Pick the monomial $F_{k_1} \ldots F_{k_m}$ in r with largest total size $k_1 + \ldots + k_m$, and amongst those pick one with m largest. Now expanding r in terms of the w_α basis, we see that $\tilde{R}_{(k_1, \ldots, k_m)}$ cannot come from any other monomial in r, giving a contradiction. $\square$

8.2. **The duality between** $\mathfrak{M}\mathrm{NSym}$ **and** $\mathfrak{m}\mathrm{QSym}$.

Theorem 8.4. *The subspace* $\mathfrak{M}\mathrm{NSym}$ *is a Hopf subalgebra of* $\mathfrak{M}\mathrm{MR}$ *(continuously) dual to* $\mathfrak{m}\mathrm{QSym}$, *and the basis* $\{\tilde{R}_\alpha\} \subset \mathfrak{M}\mathrm{NSym}$ *is dual to* $\{L_\alpha\} \subset \mathfrak{m}\mathrm{QSym}$.

Proof. From the definitions it is clear that the subspace $\mathfrak{M}\mathrm{NSym} \subset \mathfrak{M}\mathrm{MR}$ is the annihilator of the space I of Lemma 5.1. Since I is a biideal, it follows immediately from Theorem 7.12 that $\mathfrak{M}\mathrm{NSym}$ is a closed under product and coproduct. Thus $\mathfrak{M}\mathrm{NSym}$ is a Hopf subalgebra dual to $\mathfrak{m}\mathrm{QSym}$. One then checks that the pairing $\langle ., . \rangle : \mathfrak{M}\mathrm{MR} \otimes \mathfrak{m}\mathrm{MR} \to \mathbb{Z}$ satisfies $\langle \tilde{R}_\alpha, u + I \rangle = \delta_{\alpha\beta}$ where u is any $\mathfrak{m}$-permutation satisfying $\mathrm{Des}(u) = \beta$. $\square$

Proposition 8.5. *The Hopf algebras* $\mathfrak{M}\mathrm{NSym}$ *and* NSym *are isomorphic via the map* $F_k \mapsto S_k$.

Proof. By Proposition 8.3 the map $F_k \mapsto S_k$ is an algebra isomorphism. It remains to verify (see also Section 2.3) that $\triangle F_k = \sum_{0 \leq j \leq k} F_j \otimes F_{k-j}$, where $F_0 := 1$. This is immediate from the definition $F_k = [1, 2, \ldots, k]$. $\square$

9. The big Hopf algebra of Multi-symmetric functions

We now define the big Hopf algebra of Multi-symmetric functions $\mathfrak{M}\mathrm{Sym}$. As we will see $\mathfrak{M}\mathrm{Sym}$ is isomorphic to Sym as a Hopf algebra, but it is naturally equipped with a basis $\{g_\lambda\}$ distinct from the Schur basis $\{s_\lambda\}$.

9.1. **Reverse plane partitions.** Let λ be a partition which we associate with its Young diagram in English notation. A *reverse plane partition* T of shape λ is a filling of the boxes in λ with positive integers so that the numbers are weakly increasing in rows and columns. We write $\mathrm{sh}(T) = \lambda$ for the shape of a reverse plane partition T. For a plane partition T, define $x^T := \prod_{i \in \mathbb{P}} x_i^{T(i)}$ where $T(i)$ is *the number of columns* in T containing one or more entries equal to i. Note that

this is not the usual weight assigned to a reverse plane partition. Now define the *dual stable Grothendieck polynomials* $g_\lambda(x_1, x_2, \ldots) \in \mathbb{Z}[[x_1, x_2, \ldots]]$ by

$$g_\lambda(x_1, x_2, \ldots) = \sum_{\text{sh}(T)=\lambda} x^T.$$

Similarly define the skew polynomials $g_{\lambda/\mu}$ by taking the sum over reverse plane partitions of shape λ/μ.

Theorem 9.1. *The formal power series $g_{\lambda/\mu}(x_1, x_2, \ldots)$ are symmetric functions.*

Example 9.2. Using the definition one computes

$$g_{(2,1)} = m_{(2,1)} + 2m_{(1,1,1)} + m_{(2)} + m_{(1,1)} = s_{(2,1)} + s_{(2)}.$$

Here the m_λ's are monomial symmetric functions, see [17].

We give two proofs of this fact. Note that the top homogenous component of g_λ is just the Schur function s_λ and thus is symmetric.

Remark 9.3. The dual stable Grothendieck polynomials g_λ are implicitly studied by Lenart [10]. Shimozono and Zabrocki give a determinantal formula for the g_λ in [15].

9.2. **Fomin-Greene operators again.** Let $\mathbb{Z}\Lambda$ be the free $\mathbb{Z}$-module of formal linear combinations of partitions. For each $i \in \mathbb{P}$, we define linear operators $u_i : \mathbb{Z}\Lambda \to \mathbb{Z}\Lambda$ called *column adding operators* as follows:

$$u_i(\lambda) = \sum \mu,$$

where the sum is over all valid Young diagrams μ obtained from λ by adding several (at least one) cells to the i-th column. If no boxes can be added to the i-th column of λ then $u_i(\lambda) = 0$.

Example 9.4. If $\lambda = (4, 3, 3, 1)$ then $u_6(\lambda) = 0$, $u_5(\lambda) = (5, 3, 3, 1)$, $u_4(\lambda) = (4, 4, 3, 1) + (4, 4, 4, 1)$, $u_3(\lambda) = 0$, $u_2(\lambda) = (4, 3, 3, 2)$, $u_1(\lambda) = (4, 3, 3, 1, 1) + (4, 3, 3, 1, 1, 1) + \ldots$. Note that $u_1(\lambda)$ is always an infinite expression.

In [4] Fomin and Greene prove the following statement.

Lemma 9.5. [4, Lemma 3.1] *Assume that a set of elements $\{u_i \mid i \in \mathbb{Z}\}$ of an associative algebra satisfy the relations*

$$
\begin{aligned}
u_i u_k u_j &= u_k u_i u_j & &\text{for } i < j < k \\
u_j u_i u_k &= u_j u_k u_i & &\text{for } i < j < k \\
u_j u_i (u_i + u_j) &= (u_i + u_j) u_j u_i & &\text{for } i < j.
\end{aligned}
$$

Then the noncommutative analogs of elementary symmetric functions

$$e_k(u_1, u_2, \ldots) = \sum_{a_1 > a_2 > \ldots > a_k} u_{a_1} u_{a_2} \ldots u_{a_k}$$

commute.

Note that the statement of Lemma 9.5 is formal: the $e_k(u_1, u_2, \ldots)$ should be considered as elements of an appropriate completion.

Lemma 9.6. *The column adding operators $u_i : \mathbb{Z}\Lambda \to \mathbb{Z}\Lambda$ satisfy the relations in Lemma 9.5.*
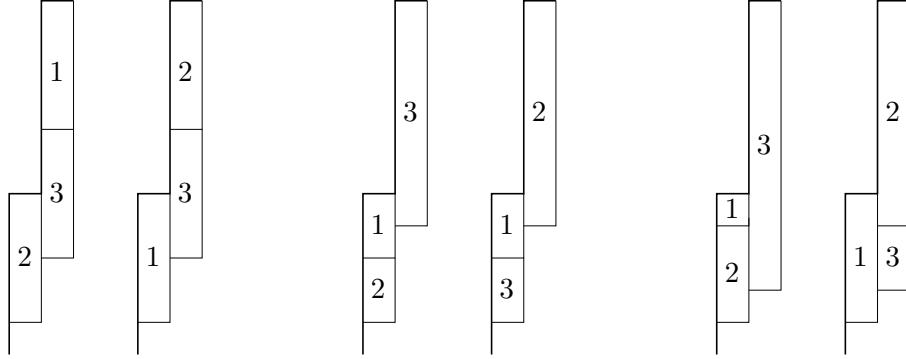
FIGURE 1. The bijection between terms occurring in $u_j u_i (u_i + u_j)(\lambda)$ and in $(u_i + u_j) u_j u_i(\lambda)$. The three possible cases are shown.

Proof. The first two relations are straightforward since if $|i - k| > 1$ operators u_i and u_k can easily be seen to commute.

Thus, it remains to argue that the third relation holds. Again, if $|i - j| > 1$ operators u_i and u_j commute and the relation follows. Thus the only non-trivial case is $j = i + 1$. Let μ be a partition occurring with non-zero coefficient in $u_j u_i (u_i + u_j)(\lambda)$. Then μ can be obtained from λ in the following ways, where by $\lambda +_i a$ we denote operation of adding $a \geq 1$ cells to i-th column of λ.

(1) $\mu = \lambda +_j a' +_i b +_j a''$;
(2) $\mu = \lambda +_i b' +_i b'' +_j a$ and $\lambda +_i b' +_j a +_i b''$ is a valid sequence;
(3) $\mu = \lambda +_i b' +_i b'' +_j a$ and $\lambda +_i b' +_j a +_i b''$ is not a valid sequence.

Thus μ differs from λ in the i-th column by $b = b' + b''$ squares and in the $(i+1)$-th column by $a = a' + a''$ squares. We now show how the three cases above correspond to terms equal to μ occurring in $(u_i + u_j) u_j u_i(\lambda)$.

In case (1), we rearrange the terms to get $\mu = \lambda +_i b +_j a' +_j a''$, which must be a valid sequence. In case (2), we biject the expression $\lambda +_i b' +_i b'' +_j a$ with the valid sequence $\lambda +_i b' +_j a +_i b''$. Finally, in case (3), we biject $\lambda +_i b' +_i b'' +_j a$ with the sequence $\lambda +_i b +_j a' +_j a''$, where $a' < a$ is the maximal number such that $\lambda +_i b' +_j a'$ is a valid sequence, and $a'' = a - a'$. Note that $a'' \geq b' \geq 1$.

One can verify now that we get each possible summand of $(u_i + u_j) u_j u_i(\lambda)$ equal to μ exactly once this way. Indeed, we have the following three cases:

(1) $\mu = \lambda +_i b +_j a' +_j a''$ where $\mu = \lambda +_j a' +_i b +_j a''$ is a valid sequence;
(2) $\mu = \lambda +_i b' +_j a +_i b''$;
(3) $\mu = \lambda +_i b +_j a' +_j a''$ where $\mu = \lambda +_j a' +_i b +_j a''$ is not a valid sequence.

These three cases correspond exactly to the three cases listed before.

An illustration of the proof is given in Figure 1.

$\square$

Proof of Theorem 9.1. Define the formal power series $A(x)$ with coefficients in operators on $\mathbb{Z}\Lambda$ by $A(x) = \cdots (1 + xu_2)(1 + xu_1)$. Lemma 9.5 and Lemma 9.6 essentially says that $A(x)A(y) = A(y)A(x)$ for two formal variables x and y. It is
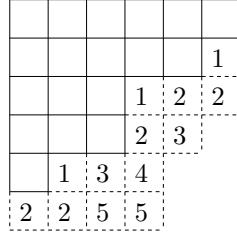
FIGURE 2. An elegant filling for $\lambda = (6, 6, 6, 5, 4, 4)$ and $\mu = (6, 5, 3, 3, 1)$.

immediate from the definition that

$$g_{\lambda/\mu}(x_1, x_2, \ldots) = \langle \cdots A(x_2)A(x_1)\mu, \lambda \rangle$$

where $\langle ., . \rangle : \mathbb{Z}\Lambda \otimes \mathbb{Z}_{\text{fin}}\Lambda \to \mathbb{Z}$ is the pairing defined by $\langle \nu, \rho \rangle = \delta_{\rho\nu}$, and $\mathbb{Z}_{\text{fin}}\Lambda$ denotes the free $\mathbb{Z}$-module of *finite* linear combinations of partitions. Since the $A(x_i)$ commute, $g_{\lambda/\mu}(x_1, x_2, \ldots)$ is a symmetric function in the variables x_i. $\square$

Remark 9.7. It would be interesting to compare the operators u_i of this section with the operators v_i of Section 6. As we shall see the two kinds of operators are dual in a sense which will be made clear in Theorem 9.15.

9.3. **Schur decomposition of dual stable Grothendieck polynomials.** We give a direct bijection to establish a stronger version of Theorem 9.1. Namely, we describe an explicit rule for the decomposition of the g_λ's into Schur functions.

Given two partitions λ and μ define the number f_λ^μ as follows. If $\mu \nsubseteq \lambda$, we set $f_\lambda^\mu = 0$. Otherwise, f_λ^μ is equal to the number of *elegant fillings* of the skew shape λ/μ. A filling is elegant if it satisfies the following two conditions:

(1) it is semistandard - that is, the numbers weakly increase in rows and strictly in columns, and
(2) the numbers in row i lie in $[1, i - 1]$.

In particular there is no elegant filling of λ/μ if it contains a cell in first row. Elegant fillings were used previously in [10]. An example of an elegant filling is given in Figure 2.

Theorem 9.8. *Let λ be a partition. Then $g_\lambda(x_1, x_2, \ldots) = \sum_\mu f_\lambda^\mu s_\mu(x_1, x_2, \ldots)$.*

Proof. We construct a weight preserving bijection between reverse plane partitions T of shape λ and pairs (S, U), where S is a semistandard tableau of shape μ and U is an elegant filling of shape λ/μ. Assume λ has m rows, and denote by T_i the i-th row of T. More generally, let $T_{[i,j]}$ be the reverse plane partition consisting of the part of T between rows i and j inclusively. For each $i \in [1, m]$, define the *reduction* $\tilde{T}_i$ of the row T_i to be the sequence of numbers obtained from T_i by removing all entries equal to the corresponding entries in T_{i+1} immediately below.

To define the bijection we proceed recursively, defining a sequence (S_i, U_i) such that S_i is a semistandard tableaux satisfying $\text{sh}(S_i) \subset (\lambda_i, \lambda_{i+1}, \ldots, \lambda_m)$ and U_i is an elegant filling of shape $(\lambda_i, \lambda_{i+1}, \ldots, \lambda_m)/\text{sh}(S_i)$. For the first step, set $S_m := T_m$ and U_m to be the (empty) elegant filling of $(\lambda_m)/(\lambda_m)$. Assume now we have defined (S_{k+1}, U_{k+1}). We use the Robinson-Schensted-Knuth (RSK) algorithm (see [17])
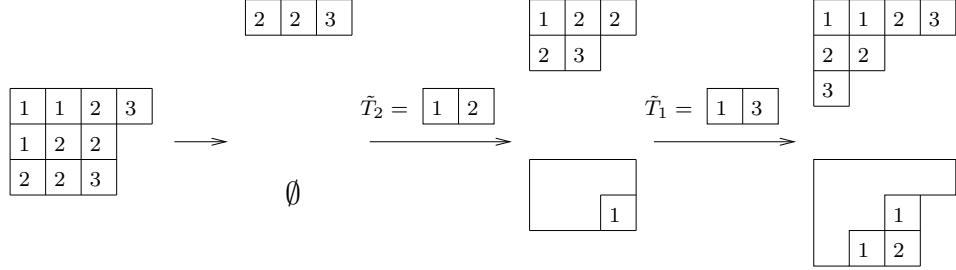
FIGURE 3. The bijection in the proof of Theorem 9.8.

to insert the reduced row $\tilde{T}_k$ into S_{k+1}, obtaining S_k. We verify by induction that

(5)   the first row of S_i will always coincide with T_i.

Indeed, this is true for S_m, and the insertion of $\tilde{T}_i$ will always push out from the first row of S_{i+1} elements of T_{i+1} which are strictly greater than elements of T_i immediately above them inside T.

Now we describe how to obtain U_k. First shift all the numbers in U_{k+1} one row down and simultaneously adding 1 to each of them, and consider the result as a partial filling of the cells of the skew shape $\tau = (\lambda_k, \lambda_{k+1}, \ldots, \lambda_m)/\operatorname{sh}(S_{k+1})$. Note that the unfilled cells of τ form a horizontal strip H of length λ_k. Now by well-known properties of RSK, the difference $\operatorname{sh}(S_k)/\operatorname{sh}(S_{k+1})$ is a horizontal strip of size $|\tilde{T}_k| \leq \lambda_k$, and this horizontal strip is contained in H. We obtain U_k by placing a 1 in every cell that lies in H but not in $\operatorname{sh}(S_k)/\operatorname{sh}(S_{k+1})$, thus obtaining a filling of shape $(\lambda_k, \lambda_{k+1}, \ldots, \lambda_m)/\operatorname{sh}(S_k)$. Since the 1's in U_k form a horizontal strip, and by assumption U_{k+1} is semistandard, we conclude that U_k is semistandard. Also all the entries in U_{k+1} are moved one row down and incremented, and by (5) none of the new 1's are placed in the first row so U_k must be elegant, again assuming that U_{k+1} was elegant. Proceeding in this manner we obtain a pair $(S, U) := (S_1, U_1)$.

Figure 3 illustrates this direction of the bijection.

To prove that the defined map is a bijection we describe the inverse map. Start with a pair $(S, U) = (S_1, U_1)$ where S is a semistandard tableau of shape μ and U is an elegant filling of shape λ/μ. Construct recursively a sequence of pairs (S_i, U_i) such that U_i has shape $\nu^{(i)}/\operatorname{sh}(S_i)$ for some $\nu^{(i)}$, as follows. Assume (S_k, U_k) has already been constructed. Define the *boundary* of S_k to be the set of cells in S_k directly below which there are no other cells. The boundary is a horizontal strip with size equal to the size of the first row of S_k. Define the *active boundary* of S_k to be the subset of cells of the boundary below which the cell of U_k does not contain a 1. Note that active boundary is again always a horizontal strip. Now apply the inverse RSK algorithm to the active boundary of S_k, producing a smaller semistandard tableau S_{k+1} and a non-decreasing sequence of numbers $\bar{T}_k$. In order to get U_{k+1} remove the 1's from U_k, decrease all the remaining numbers by 1 and move them one row up. It is evident that U_{k+1} is an elegant filling if U_k was. By the choice of active boundary it is also clear that the shape of U_{k+1} "fits" with the shape of S_{k+1}.

Now define T by letting its i-th row T_i equal the first row of S_i. By properties of the (inverse) RSK algorithm T must be a valid reverse plane partition. Also observe

that the sequences $\bar{T}_i$ ejected during the construction are equal to the reduced rows $\tilde{T}_i$ of T. Indeed, by the nature of inverse RSK all the elements of $\bar{T}_i$ were either bumped out from the first row of S_i by bigger numbers or belonged to the part of the active boundary in the first row of S_i. This means that the numbers $\bar{T}_i$ are exactly the elements of T_i that are not equal to the element of T_{i+1} immediately below, which is the definition of $\tilde{T}_i$. This shows that defined map is indeed a bijection.

Since S is obtained by inserting the reductions $\tilde{T}_i$ of the rows of T, we have $x^T = x^S$. Thus the bijection is weight-preserving, completing the proof. $\qquad\square$

Example 9.9. The decomposition $g_{(3,2,2)} = s_{(3,2,2)} + 2s_{(3,2,1)} + s_{(3,1,1)} + 3s_{(3,2)} + 2s_{(3,1)} + s_{(3)}$ corresponds to the following elegant fillings:
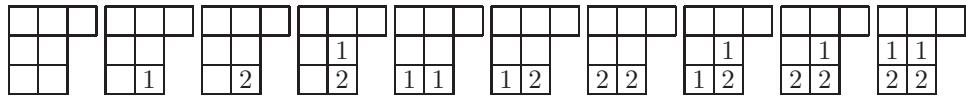
Let $\mathfrak{M}\mathrm{Sym} = \oplus_\lambda \mathbb{Z}g_\lambda$ be the free $\mathbb{Z}$-module consisting of finite $\mathbb{Z}$-linear combinations of the g_λ.

Proposition 9.10. *The elements g_λ form a basis for the ring of symmetric functions. Thus $\mathfrak{M}\mathrm{Sym} \simeq \mathrm{Sym}$.*

Proof. By Theorem 9.8, the transition matrix between the basis of Schur functions $\{s_\lambda\}$ and the set $\{g_\lambda\}$ is upper triangular. $\qquad\square$

$\mathfrak{M}\mathrm{Sym}$ inherits from Sym a Hopf algebra structure. While isomorphic, they come with different distinguished bases: $\{g_\lambda\}$ and $\{s_\lambda\}$.

Proposition 9.11. $\mathfrak{M}\mathrm{Sym}$ *is freely generated by the set $\{g_{(n)} \mid n \geq 1\}$ as an algebra.*

Proof. Since $g_{(n)} = h_n$ are exactly the complete homogenous symmetric functions, this follows from the well known fact that $\mathrm{Sym} = \mathbb{Z}[h_1, \ldots, h_n]$. $\qquad\square$

Let ρ and τ be two skew shapes. Denote by $\rho \rhd \tau$ the skew shape obtained by attaching the two so that lower leftmost cell of τ is directly to the right of the upper rightmost cell of ρ. Denote by $\rho \lhd \tau$ the skew shape obtained by attaching the two shapes so that lower leftmost cell of τ is directly above the upper rightmost cell of ρ. Finally, denote by $\rho \cdot \tau$ the skew shape obtained by attaching the two shapes so that lower leftmost cell of τ coincides with the upper rightmost cell of ρ.

Recall that in Section 2.4 we have associated a ribbon skew shape r_α to each composition α. The operations $\alpha \rhd \beta$, $\alpha \cdot \beta$ and $\alpha \lhd \beta$ in Section 8.1 are consistent with the ones introduced here:

$$r_{\alpha \rhd \beta} = r_\alpha \rhd r_\beta \quad r_{\alpha \cdot \beta} = r_\alpha \cdot r_\beta \quad r_{\alpha \lhd \beta} = r_\alpha \lhd r_\beta.$$

Lemma 9.12. *Let ρ and τ be two skew shapes. We have $g_\rho g_\tau = g_{\rho \rhd \tau} + g_{\rho \lhd \tau} - g_{\rho \cdot \tau}$.*

Proof. Let R be a reverse plane partition of shape ρ and T be a reverse plane partition of shape τ. Let R_{ur} be the label of upper rightmost cell of R and let T_{ll} be the label of the lower leftmost cell of T. If $R_{ur} \leq T_{ll}$ attach R and T by putting R_{ur} to the immediate left of T_{ll} so that we get a reverse plane partition $R \rhd T$ of shape $\rho \rhd \tau$. Note that $x^{R \rhd T} = x^R x^T$. If $R_{ur} > T_{ll}$ attach the two so that we get a reverse plane partition $R \lhd T$ of shape $\rho \lhd \tau$. Note again that $x^{R \lhd T} = x^R x^T$.
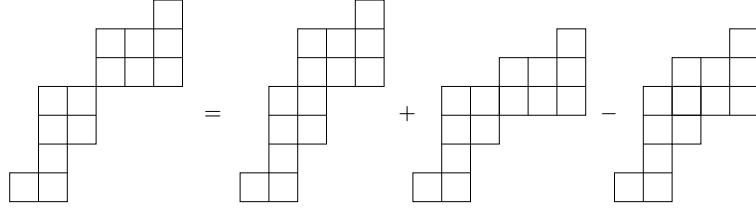
FIGURE 4. The four shapes involved in Lemma 9.12.

However, this map is not a bijection between pairs (R, T) of reverse plane partitions of shape ρ or τ and a reverse plane partition of shape $\rho \rhd \tau$ or $\rho \lhd \tau$. There are additional reverse plane partitions of shape $\rho \lhd \tau$ which cannot be obtained in this way, namely, the ones where the box corresponding to the lower leftmost box of τ has *the same* entry as the box corresponding to upper rightmost cell of ρ. Such reverse plane partitions are in (weight-preserving) bijection with reverse plane partitions of shape $\rho \cdot \tau$, which finishes the proof. $\qquad\square$

Figure 4 illustrates Lemma 9.12 by showing the four shapes involved. For each skew shape λ/μ, define the symmetric function $\tilde{g}_{\lambda/\mu}$ by

$$\tilde{g}_{\lambda/\mu}(x_1, x_2, \ldots) = (-1)^{|\lambda/\mu|} \, g_{\lambda/\mu}(-x_1, -x_2, \ldots).$$

Thus $\tilde{g}_{\lambda/\mu}$ differs from $g_{\lambda/\mu}$ by a sign in each homogeneous component.

Theorem 9.13. *The map $F_k \mapsto g_{(k)} = \tilde{g}_{(k)}$ is a surjective Hopf algebra morphism from $\mathfrak{M}$NSym to $\mathfrak{M}$Sym which sends $\tilde{R}_\alpha$ to $\tilde{g}_{r_\alpha}$ for each composition α. Thus, $\mathfrak{M}$Sym is the commutative image of $\mathfrak{M}$NSym.*

Proof. The first statement is immediate from Propositions 8.3, 8.5, 9.10 and 9.11. Since $|\tau \cdot \rho| = |\tau| + |\rho| - 1$, Lemma 9.12 implies that $\tilde{g}_\rho \tilde{g}_\tau = \tilde{g}_{\rho \rhd \tau} + \tilde{g}_{\rho \cdot \tau} + \tilde{g}_{\rho \lhd \tau}$. For ribbon shapes, this agrees with Proposition 8.2, giving the statement of the theorem. $\qquad\square$

9.4. **Duality between $\mathfrak{M}$Sym and $\mathfrak{m}$Sym.** In [10] Lenart proved the following theorem.

Theorem 9.14. [10, Theorem 2.8] *For a partition λ, one has*

$$s_\lambda = \sum_{\lambda \subset \mu} f^\lambda_\mu G_\mu$$

where f^λ_μ is the number of elegant fillings of μ/λ.

Using Theorem 9.14, we can relate the g_λ to G_λ.

Theorem 9.15. *The Hopf algebras $\mathfrak{M}$Sym and $\mathfrak{m}$Sym are continuously dual Hopf algebras via the Hall inner product. The bases $\{g_\lambda \mid \lambda \in \Lambda\}$ and $\{G_\lambda \mid \lambda \in \Lambda\}$ are dual bases. The bases $\{\tilde{g}_\lambda \mid \lambda \in \Lambda\}$ and $\{\tilde{K}_\lambda \mid \lambda \in \Lambda\}$ are dual bases.*

Proof. By Proposition 9.10, $\mathfrak{M}$Sym is isomorphic to Sym, and by Proposition 6.4, $\mathfrak{m}$Sym is isomorphic to the completion of Sym. Since Sym is self dual under the Hall inner product, $\mathfrak{M}$Sym and $\mathfrak{m}$Sym are continuously dual with this pairing.

But Theorem 9.14 with Theorem 9.8, one immediately concludes that $\{G_\lambda \mid \lambda \in \Lambda\}$ and $\{g_\lambda \mid \lambda \in \Lambda\}$ are dual bases. To obtain the last statement we use the isomorphism $f(x_1, x_2, \ldots) \mapsto f(-x_1, -x_2, \ldots)$ of Sym. $\qquad\square$

9.5. K-homology of Grassmannians.

Theorem 9.15 allows us to interpret the algebra $\mathfrak{M}$Sym as the K-homology of Grassmannians. While K-homology and K-cohomology are isomorphic for Grassmannians, we will find that $\mathfrak{M}$Sym is functorially covariant, like K-homology, while $\mathfrak{m}$Sym is contravariant in the corresponding sense. We use the notation introduced in Section 6 and refer the reader to [3] for further details.

There is a pairing of $K^\circ \mathrm{Gr}(k, \mathbb{C}^n)$ and $K_\circ \mathrm{Gr}(k, \mathbb{C}^n)$ obtained by the sequence $K^\circ \mathrm{Gr}(k, \mathbb{C}^n) \otimes K_\circ \mathrm{Gr}(k, \mathbb{C}^n) \to K_\circ \mathrm{Gr}(k, \mathbb{C}^n) \to K_\circ(*) = \mathbb{Z}$, where the first map is induced by taking tensor products and the second map is the pushforward to a point. If $\alpha \in K^\circ \mathrm{Gr}(k, \mathbb{C}^n)$ and $\beta \in K_\circ \mathrm{Gr}(k, \mathbb{C}^n)$ we let (α, β) denote this pairing.

Let $[\mathcal{I}_\lambda] \in K_\circ \mathrm{Gr}(k, \mathbb{C}^n)$ denote the class of the *ideal sheaf* of the boundary of the Schubert variety X_λ. For $\lambda \subset (n-k)^k$, we let $\tilde{\lambda} = (n-k-\lambda_k, \ldots, n-k-\lambda_1)$ denote the rotated complement of λ in the $(n-k)^k$ rectangle. Buch shows in [3, p.30] that the classes $[\mathcal{I}_\lambda]$ form a basis dual to the classes $[\mathcal{O}_\lambda]$ of structure sheaves of Schubert varieties. More precisely, one has $([\mathcal{O}_\lambda], [\mathcal{I}_{\tilde{\mu}}]) = \delta_{\lambda\mu}$.

Via Theorem 6.6 we may identify the limit of this pairing as $k, n \to \infty$ with the Hall inner product. In this way one may identify quotients of $\mathfrak{M}$Sym with the K-homologies $K_\circ \mathrm{Gr}(k, \mathbb{C}^n)$ of Grassmannians, as we now explain. Again for convenience we let $[\mathcal{I}_{\tilde{\lambda}}] = 0$ if λ does not fit in a $(n-k)^k$ rectangle.

Theorem 9.16. *The map $\mathfrak{M}$Sym $\to K_\circ \mathrm{Gr}(k, \mathbb{C}^n)$ given by $g_\lambda \mapsto [\mathcal{I}_{\tilde{\lambda}}]$ is a surjection. It identifies the comultiplication of $\mathfrak{M}$Sym with the map*

$$\Delta_* : K_\circ \mathrm{Gr}(k, \mathbb{C}^n) \to K_\circ \mathrm{Gr}(k, \mathbb{C}^n) \otimes K_\circ \mathrm{Gr}(k, \mathbb{C}^n)$$

induced by the diagonal embedding $\Delta : \mathrm{Gr}(k, \mathbb{C}^n) \to \mathrm{Gr}(k, \mathbb{C}^n) \times \mathrm{Gr}(k, \mathbb{C}^n)$ and the multiplication of $\mathfrak{M}$Sym with the maps

$$\phi_* : K_\circ \mathrm{Gr}(k_1, \mathbb{C}^{n_1}) \otimes K_\circ \mathrm{Gr}(k_2, \mathbb{C}^{n_2}) \to K_\circ \mathrm{Gr}(k_1 + k_2, \mathbb{C}^{n_1 + n_2})$$

induced by $\phi : \mathrm{Gr}(k_1, \mathbb{C}^{n_1}) \times \mathrm{Gr}(k_2, \mathbb{C}^{n_2} \to \mathrm{Gr}(k_1 + k_2, \mathbb{C}^{n_1 + n_2})$ (see discussion after Theorem 6.6).

Proof. The first statement is clear from the definitions since the classes $\{[\mathcal{I}_{\tilde{\lambda}}] \mid \lambda \subset (n-k)^k\}$ of the ideal sheaves form a basis $K_\circ \mathrm{Gr}(k, \mathbb{C}^n)$. We will check the "comultiplication" statement (the last statement is similar). Let $X = \mathrm{Gr}(d, \mathbb{C}^n)$. The product of two classes in $K^\circ X$ can be calculated via the pullback $\Delta^* : K^\circ X \otimes K^\circ X \to K^\circ X$ in K-theory: $\Delta^*([\mathcal{O}_\lambda] \otimes [\mathcal{O}_\mu]) = [\mathcal{O}_\lambda].[\mathcal{O}_\mu]$. By Theorem 6.6, the coefficient of G_ν in $G_\lambda G_\mu$ is equal to the coefficient of $[\mathcal{O}_\nu]$ in $[\mathcal{O}_\lambda].[\mathcal{O}_\mu]$. This in turn can be calculated via the projection formula as $(\Delta^*([\mathcal{O}_\lambda] \otimes [\mathcal{O}_\mu]), [\mathcal{I}_{\tilde{\nu}}]) = ([\mathcal{O}_\lambda] \otimes [\mathcal{O}_\mu], \Delta_*[\mathcal{I}_{\tilde{\nu}}]) = \sum_{\rho\tau} a_\nu^{\rho\tau}([\mathcal{O}_\lambda], [\mathcal{I}_{\tilde{\rho}}])([\mathcal{O}_\mu], [\mathcal{I}_{\tilde{\tau}}]) = a_\nu^{\rho\tau}$ where $\Delta_*[\mathcal{I}_{\tilde{\nu}}] = \sum_{\rho\tau} a_\nu^{\rho\tau}[\mathcal{I}_{\tilde{\rho}}] \otimes [\mathcal{I}_{\tilde{\tau}}]$. By Theorem 9.15, the product structure constants for $\{G_\lambda \mid \lambda \in \Lambda\}$ agree with the coproduct structure constants for $\{g_\lambda \mid \lambda \in \Lambda\}$. We conclude that the comultiplication Δ_* of $K_\circ X$ agrees with the comultiplication of $\mathfrak{M}$Sym. $\qquad\square$

9.6. Conjugate Fomin-Greene operators.

Let $\{u_i \mid i \in \mathbb{Z}\}$ be a set of operators satisfying Lemma 9.5. Recall that we have defined formal power series

$$A(x) = \cdots (1 + xu_1)(1 + xu_0)(1 + xu_{-1}) \cdots = \sum_{k \geq 0} e_k(u) x^k.$$

The $e_k(u)$'s commute and thus generate a homomorphic image $\mathrm{Sym}(u)$ of the algebra of symmetric functions. This allows to define $f(u)$ for any symmetric function f as follows: it is the image of $f \in \mathrm{Sym}$ under the map $\mathrm{Sym} \to \mathrm{Sym}(u)$ given by $e_k \mapsto e_k(u)$.

Define the formal power series

$$B(x) = \cdots \frac{1}{1 - xu_{-1}} \frac{1}{1 - xu_0} \frac{1}{1 - xu_1} \cdots$$

where as usual x is a formal variable commuting with all the u_i.

Lemma 9.17. *We have $B(x) = \sum_{k \geq 0} h_k(u)x^k$ where $h_0(u) = 1$.*

Proof. For each $l \geq 0$ we have the well known identity $\sum_{k=0}^{l}(-1)^k h_k e_{n-k} = 0$. From this one deduces that $A(x)\tilde{B}(x) = 1$, where $\tilde{B}(x) = \sum_{k \geq 0} h_k(u)x^k$. On the other hand $A(x)B(x) = 1$ also holds. This implies $\tilde{B}(x) = B(x)$. □

Lemma 9.18. *We have*

$$\cdots A(x_2)A(x_1) = \sum_{\lambda} s_\lambda(u)s_{\lambda'}(x)$$

and

$$\cdots B(x_2)B(x_1) = \sum_{\lambda} s_\lambda(u)s_\lambda(x)$$

where the sums are over all partitions λ.

Proof. Start with the usual Cauchy identity

$$\prod_{i,j=1}^{\infty} (1 + x_i y_j) = \sum_{\lambda} s_\lambda(y)s_{\lambda'}(x).$$

Group the terms on the left hand side so that we get

$$\prod_{j=1}^{\infty} \sum_{k=0}^{\infty} e_k(y)x_j^k = \sum_{\lambda} s_\lambda(y)s_{\lambda'}(x).$$

Now apply the transformation $\mathrm{Sym} \to \mathrm{Sym}(u)$ given by $e_k(y) \mapsto e_k(u)$ to both sides. We get exactly the first equality. The proof of the second one is analogous. □

Now let us assume that the operators u_i act on the space $\mathbb{Z}\Lambda$ of formal $\mathbb{Z}$-linear combinations of all partitions. As before, we define the inner product $\langle \lambda, \mu \rangle = \delta_{\lambda\mu}$. Define $M_{\mu/\nu} = \langle \cdots A(x_2)A(x_1) \cdot \nu, \mu \rangle$ and $N_{\mu/\nu} = \langle \cdots B(x_2)B(x_1) \cdot \nu, \mu \rangle$. Since the u_i satisfy the conclusion of Lemma 9.5, both $M_{\mu/\nu}$ and $N_{\mu/\nu}$ are symmetric functions in the variables $x_1, x_2, \ldots$.

Lemma 9.19. *For a set of operators (or noncommutative variables) $\{u_i \mid i \in \mathbb{Z}\}$ satisfying the conditions of Lemma 9.5 we have $\omega(M_{\mu/\nu}) = N_{\mu/\nu}$, where $\omega : \mathrm{Sym} \to \mathrm{Sym}$ is the algebra involution given by $e_k \mapsto h_k$.*

Proof. We compute

$$\omega(M_{\mu/\nu}) = \omega\left(\sum_{\lambda} \langle s_\lambda(u) \cdot \nu, \mu \rangle s_{\lambda'}(x) \right) = \sum_{\lambda} \langle s_\lambda(u) \cdot \nu, \mu \rangle s_\lambda(x) = N_{\mu/\nu}.$$

□

9.7. **Weak set-valued tableaux.** Fix a partition ν. Now we specialize the situation in Section 9.6 to the operators $\{v_i \mid i \in \mathbb{Z}\}$ defined in Section 6 which act on $\mathbb{Z}\Lambda_\nu$. The action of the product $1/(1-xv_1)1/(1-xv_2)\cdots$ on $\mathbb{Z}\Lambda_\nu$ then corresponds to the following *weak set-valued tableaux*.

Definition 9.20. A *weak set-valued tableau T* of shape λ/ν is a filling of the boxes with finite non-empty multisets of positive integers (thus, numbers in one box are not necessarily distinct) so that

(1) the smallest number in each box is strictly bigger than the largest number in the box directly to the left of it (if that box is present);

(2) the smallest number in each box is greater than or equal to the largest number in the box directly above it (if that box is present).

For a *weak set-valued tableau T*, define x^T to be $\prod_{i \geq 1} x_i^{a_i}$ where a_i is the number of occurrences of the letter i in T.

This differs from the set-valued tableaux of Buch [3] in two ways: (a) the strict and weak inequalities have been swapped, and (b) repeated numbers are allowed in each box. The following weak set-valued tableau T has weight $x^T = x_1 x_2^3 x_3 x_4^3 x_5^2$.

12	44
223	5
45	

Let $J_{\lambda/\nu} = \sum_T x^T$ denote the weight generating function of all weak set-valued tableaux T of shape λ/ν.

Theorem 9.21. *We have $J_{\lambda/\nu}(x_1, x_2, \ldots) = \langle \cdots B(x_2)B(x_1) \cdot \nu, \lambda \rangle$ where*

$$B(x) = \cdots \frac{1}{1-xv_{-1}} \frac{1}{1-xv_0} \frac{1}{1-xu_1} \cdots .$$

In particular, $J_{\lambda/\nu}(x_1, x_2, \ldots)$ is a symmetric function in the variables $x_1, x_2, \ldots$.

Proof. The result is established in the same way as [3, Theorem 3.1]. The multiple occurences of a single number in a box correspond to the degree 2 and higher terms of the expansion $1/(1-xv_i) = 1 + xv_i + x^2v_i^2 + \cdots$. The reversal of the order of operators changed the strict and weak inequalities; or in other words, swapped the notions of horizontal and vertical strips. $\square$

The following is a direct consequence of (the proof of) Lemma 9.19, Lemma 6.1 and Theorem 9.21.

Proposition 9.22. *For any skew shape λ/ν, we have $\omega(\tilde{K}_{\lambda/\nu}) = J_{\lambda/\nu}$.*

9.8. **Valued-set tableaux.** Now let $u_i : \mathbb{Z}\Lambda \to \mathbb{Z}\Lambda$ be the operators defined in Section 9.2.

Definition 9.23. A *valued-set tableaux T* of shape λ/μ is a filling of the boxes of λ/μ with positive integers so that

(1) the transpose of this filling of T is a (usual) semistandard tableau, and

(2) we are provided with the additional information of a decomposition of the shape into a disjoint union $\lambda/\mu = \sqcup A_j$ of groups A_j of boxes so that each A_i is connected and completely contained within a single column and all boxes in each A_i contains the same number.

For a *valued-set tableau* T, define x^T to be $\prod_{i \geq 1} x_i^{a_i}$ where a_i is the number of groups A_j which contain the letter i.

An example of a valued-set tableau can be seen in Figure 5. The grouping of the boxes is shown by omitting the edge separating the boxes.

1	2	4	5
1	3	4	6
1	3	5	6
2	3		
2	3		
3			

FIGURE 5. An example of a valued-set tableau T with shape $(4, 4, 4, 2, 2, 1)$. We have $x^T = x_1^2 x_2^3 x_3^3 x_4 x_5^2 x_6$.

Let $j_{\lambda/\mu} = \sum_T x^T$ denote the generating function of all valued-set tableaux of shape λ/μ.

Theorem 9.24. *We have* $j_{\lambda/\nu}(x_1, x_2, \ldots) = \langle \cdots B(x_2) B(x_1) \cdot \nu, \lambda \rangle$ *where*

$$B(x) = \cdots \frac{1}{1 - x u_0} \frac{1}{1 - x u_1} \frac{1}{1 - x u_2} \cdots .$$

In particular, $j_{\lambda/\nu}(x_1, x_2, \ldots)$ is a symmetric function in the variables $x_1, x_2, \ldots$.

Proof. The operator $B(x)$ acting on a partition ν adds a vertical strip. If μ/ν is a vertical strip, the coefficient x^k in $\langle B(x) \cdot \nu, \mu \rangle$ is the number of ways to write each column of μ/ν as a disjoint union of non-empty groups of boxes, using k groups in total. This recovers the definition of a valued-set tableau. The last statement follows from $B(x)B(y) = B(y)B(x)$.   □

The following is a direct consequence of Lemma 9.19 and Theorem 9.21.

Proposition 9.25. *We have* $\omega(g_{\lambda/\mu}) = j_{\lambda/\mu}$.

Note that since $\omega : \mathrm{Sym} \to \mathrm{Sym}$ is an algebra automorphism, the K-theory and K-homology of Grassmanians can also be described in terms of J_λ's and j_λ's.

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